Exercise 1. ***Pendulum large oscillations***

Given a simple pendulum of mass m and length l. Assume that, at the initial time, the pendulum has zero velocity and it forms an angle ϕ_0 with respect to the vertical axis. Do not assume, for the moment, small oscillations.

a) Determine the period of oscillation as a function of its amplitude.

It is not required to explicitly perform the integral: you should be able to express it as a *complete elliptic integral of the first kind*

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}} . \tag{1}$$

Hint: use the fact that $(1 - \cos\phi)/2 = \sin^2(\phi/2)$ *first and then the substitution* $\sin\xi = \sin\frac{\phi}{2} \,/\, \sin\frac{\phi_0}{2}$ *, where* ϕ *is the angle the pendulum forms with the vertical axis.*
What is the difference between this case and the case of small oscillations seen in the third exercise sheet?

b) Knowing that the integral defined above has the following Taylor expansion

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2\,n)!}{2^{2n}\,(n!)^2} \right]^2 k^{2n} , \tag{2}$$

express the period of the pendulum as a power expansion up to the fourth power of the amplitude. Do you recognise any of the terms appearing?

Exercise 2. ***Rotating bead on a plane***

A bead of mass m is constraint to move on a horizontal disc placed at height H from the ground. Now we puncture a small hole at the center of the disc, we attach a massless wire of length L to the bead and we pass it through the hole; when the bead is in the rest position at the center of the disc, then, the wire completely hangs loose vertically. In order to describe the motion of bead use cylindrical coordinates (ρ, ϕ). We look at the following two cases:

(a) We attach a weight of mass M and height $H - L$ to the other end of the wire, in a way that the weight touches the ground and the wire is completely straight above it. Now we pull the bead horizontally, such that the weight reaches a height of z from the ground, and we give a spin to the bead, placing it in an (instantaneous) circular motion with angular velocity ω (see Fig 1).

  (i) Derive the Lagrangian for the bead in this configuration. Neglect the friction in your derivation.

  (ii) Write down the Euler-Lagrange equations, and using one of the two show that the angular momentum l of the bead is a conserved quantity.

(b) Now we remove the weight attached to the wire and we replace it with a spring fixed on the ground, which has a spring constant k and rest position at $z_0 = H - L$. Similarly as before, we pull the bead horizontally until the spring is stretched to a height z, and then we let it spin on the disc with angular velocity ω (see Fig 2).

(i) Derive the Lagrangian for the bead in this other configuration. Neglect the friction.

(ii) Write down the Euler-Lagrange equations and show that the angular momentum l of the bead is conserved.

(c) In both cases cast the Euler-Lagrange equations into the following form:

$$m\ddot{\rho} = -\frac{d}{d\rho}\left(U_{\text{eff.}}\right), \tag{3}$$

where $U_{\text{eff.}}(\rho) = U(\rho) + \frac{1}{2}\frac{l^2}{m\rho^2}$. Show that for both cases there is an orbit which is restricted to some region $[\rho_{\min}, \rho_{\max}]$.

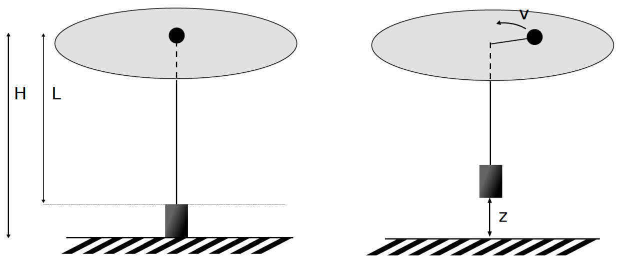

Figure 1: Graphical depiction of the system for exercise 2a).

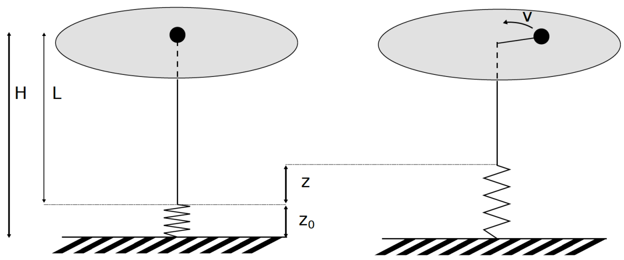

Figure 2: Graphical depiction of the system for exercise 2b).

Exercise 3. *The variation of the energy of a holonomic system*

a) Write the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j(\dot{\vec{r}}_j)^2 \tag{4}$$

of a holonomic system of N particles as a function of n independent generalized coordinates q_i $(i = 1, \ldots, n)$ and show that for stationary constraints (i.e., constraints without explicit dependence on time)

$$f_k(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_j) = 0\,, \qquad k = 1, \ldots, d \quad \text{where } d \text{ is the number of constraints,} \tag{5}$$

T is a quadratic, homogeneous polynomial of the generalized velocities $\dot{q}_i$.

Hint. Write $T = T_2 + T_1 + T_0$ where T_2 is quadratic in $\dot{q}_i$, T_1 is linear in $\dot{q}_i$ and T_0 is independent of $\dot{q}_i$.

b) Given that the total energy of a holonomic system is $E = T + U$, where T is the kinetic energy and U is the potential energy, use the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \tilde{Q}_i \tag{6}$$

to show that

$$\frac{dE}{dt} = \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial T}{\partial t} + \frac{\partial U}{\partial t} + \sum_i \tilde{Q}_i \dot{q}_i \,, \tag{7}$$

where $\tilde{Q}_i$ is a non-potential generalized force.

Hint. Make use of the Euler formula for a homogeneous polynomial $f = f(x_1, \ldots, x_k)$ of degree m,

$$\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} x_i = mf \,. \tag{8}$$

c) Consider a few cases of the result (7).

(i) system with stationary constraint

(ii) the potential energy does not depend explicitly on time and (i)

(iii) a conservative system and (ii)

d) Non-potential forces $\tilde{Q}_i$ are called gyroscopic if

$$\sum_i \tilde{Q}_i \dot{q}_i = 0 \,. \tag{9}$$

Consider a generalized force which depends linearly on the generalized velocities

$$\tilde{Q}_i = \sum_j a_{ij} \dot{q}_j \,. \tag{10}$$

Under what conditions is $\tilde{Q}_i$ gyroscopic? Is the *Coriolis force* acting on a particle of mass m a gyroscopic force?

$$\vec{\tilde{Q}} = -2m\,\vec{\omega} \times \vec{v} \tag{11}$$