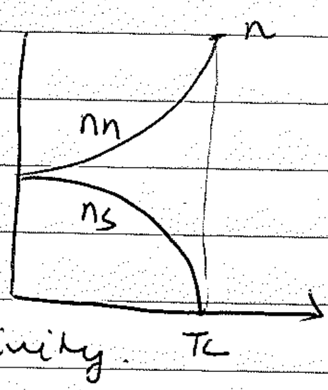
Two fluid model

- normal fluid of e^{-s} with density n_n
- superconducting (SC) fluid with density n_s

Total density $n = n_n + n_s$



- normal fluid, velocity v_n
  has finite resistivity
- SC fluid, velocity v_s, zero resistivity.

Current $\vec{j} = \vec{j_n} + \vec{j_s} = -e\left[n_n \vec{v_n} + n_s \vec{v_s}\right]$

Normal fluid is dissipative:

$$m\frac{d\vec{v_n}}{dt} = -e\vec{E} - \frac{m\vec{v_n}}{\tau}$$

In steady state, $\frac{m v_n}{\tau} = -e\vec{E} \Rightarrow \vec{j_n} = \underbrace{\frac{n_n e^2 \tau}{m}}_{\sigma} \vec{E}$

For the SC fluid, $m\frac{d\vec{v_s}}{dt} = -e\vec{E} \Rightarrow \frac{d\vec{j_s}}{dt} = \frac{n_s e^2 \vec{E}}{m}$

In the presence of an additional $\vec{B}$ field,

$$\frac{d\vec{v_s}}{dt} = -\frac{e}{m}\left[\vec{E} + \vec{v_s} \times \vec{B}\right]$$

Use $\frac{d\vec{v_s}}{dt} = \frac{\partial \vec{v_s}}{\partial t} + (\vec{v_s} \cdot \vec{\nabla})\vec{v_s}$

$$= \frac{\partial \vec{v_s}}{\partial t} + \vec{\nabla}\left(\frac{\vec{v_s} \cdot \vec{v_s}}{2}\right) - \vec{v_s} \times (\vec{\nabla} \times \vec{v_s})$$

This leads to,

$$\frac{\partial \vec{v_s}}{\partial t} + \frac{e\vec{E}}{m} + \vec{\nabla}\left(\frac{1}{2}\vec{v_s}\cdot\vec{v_s}\right) = \vec{v_s} \times \left[\vec{\nabla}\times\vec{v_s} - \frac{e\vec{B}}{m}\right] \quad -(1)$$

Taking the curl & using Faraday's law $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we can rewrite (1) as

$$\frac{\partial Q}{\partial t} = \vec{\nabla}\times(\vec{v_s}\times\vec{Q}) \quad -(2)$$

where $\vec{Q} \equiv \vec{\nabla}\times\vec{v_s} - \frac{e\vec{B}}{m}$.

Eq 2 $\Rightarrow$ if $\vec{Q} = 0$, it remains zero at all times.

Assuming $\vec{Q} = 0$ in equilibrium,

$$\underbrace{\vec{\nabla}\times\vec{v_s} = \frac{e\vec{B}}{m}}_{(3)} \Rightarrow \vec{\nabla}\times\vec{j_s} = -\frac{n_s e^2}{m}\vec{B}$$

Taking a curl & assuming $\dot{E} = \dot{D} = 0$ in the steady state & using Ampère's law, $\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$

we get $\vec{\nabla}^2\vec{B} = \lambda_L^{-2}\vec{B}$

$$\lambda_L = \sqrt{\frac{m}{n_s e^2 \mu_0}} \quad \text{(London penetration depth)}.$$

Condition $Q=0$, is really perfect diamagnetism.

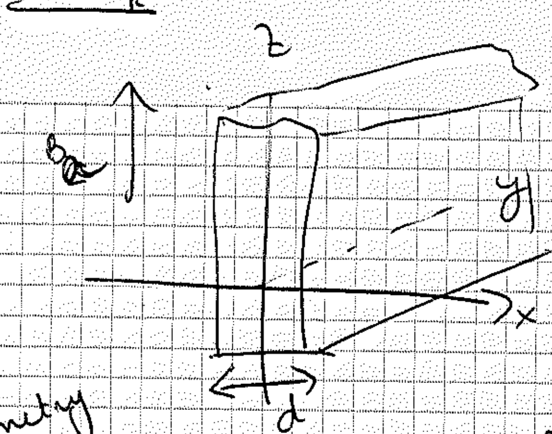
From (3)

$$\vec{\nabla}\times\frac{\vec{j_s}}{(-n_s e)} = \frac{eB}{m} \qquad \vec{\nabla}\times\vec{j_s} = \frac{-n_s e^2}{m}\vec{\nabla}\times\vec{A}$$

$$\Rightarrow \vec{j_s} = \frac{-n_s e^2}{m}\vec{A}$$

(London eqn!)

Example



$\rightarrow$ Consider the following geometry

geometry dictates $B_x = B_y = 0$

$$B_z(x) = B_1 e^{-x/\lambda_L} + B_2 e^{x/\lambda_L}$$

$$B_z(\pm \tfrac{d}{2}) = B_a$$

$$\Rightarrow \quad B_z(x) = B_a \frac{\cosh(x/\lambda_L)}{\cosh(d/\lambda_L)}$$

$d \gg \lambda_L \quad \rightarrow$ no field in bulk

$d \ll \lambda_L \quad \rightarrow$ field penetrates fully

$\parallel y$
$$J_{sy} = -\frac{B}{\mu_0 \lambda_L} \frac{\sinh(x/\lambda_L)}{\cosh(d/\lambda_L)}$$

$x = \frac{d}{2}$ $\quad J_{sy}$ out of paper

$x = -\frac{d}{2}$ $\quad$ " into paper

Magnetic field generated by such current opposes B_a
(diamagnet!)

How to measure λ_L?
measure magnetization of thin plates

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What if system was merely a perfect conductor?

$$\frac{dJ_{pc}}{dt} = E \frac{n_{pc} e^2}{m}$$

Faraday law

$$\nabla \times E = \frac{-\partial B}{\partial t}$$

$$\nabla \times \frac{\partial J_{pc}}{\partial t} = -\frac{\partial B}{\partial t} \frac{n_{pc} e^2}{m}$$ P_c = perfect cond.

For a weak diamagnet or paramagnet,

$H = \frac{B}{\mu_0}$ $\Rightarrow$ Ampère's law $\nabla \times B = \mu_0 J_{pc}$.

$$\nabla \times \left(\nabla \times \frac{\partial B}{\partial t} \right) = \mu_0 \, \nabla \times \frac{\partial J_{pc}}{\partial t}$$

$$= -\frac{\mu_0 n_{pc} e^2}{m} \frac{\partial B}{\partial t}$$

Using $\nabla \cdot B = 0$

$\longrightarrow P_C \Rightarrow \frac{\partial B}{\partial t} = 0$ in bulk rather than $B = 0$

* Penetration depth increases with temp.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{j}_{pc} = -\frac{\cancel{\vec{\nabla} \times \vec{A}}}{\Lambda} = -\frac{\vec{B}}{\Lambda}$$

$$= -\frac{n_s q^2}{m} \vec{\nabla} \times B = -\frac{n_s q^2 \mu_0}{m} \vec{j}_{pc} \Rightarrow$$

$$\nabla^2 j_s = \lambda_L^{-2} j_{pc}$$ (screening current)