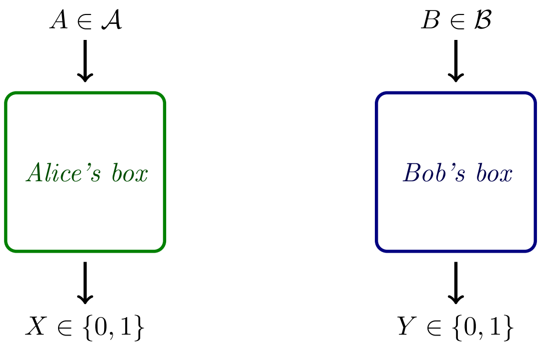
Exercise 1. *Chained Bell inequalities*

In this exercise we will encounter a Bell violation that is stronger in quantum mechanics than what we have seen so far. Let A and B denote random variables describing the input Alice and Bob give to their devices in space-like separated locations, respectively. The outputs of their devices, described by RVs X and Y, can take on values in $\{0,1\}$. Alice and Bob can choose their inputs from N different values, $A \in \mathcal{A} = \{0,2,4,\dots,2N-2\}$ and $B \in \mathcal{B} = \{1,3,5\dots,2N-1\}$.



We define I_N, a measure of correlations, by

$$I_N = P[X = Y \mid A = 0, B = 2N-1] + \sum_{|a-b|=1} P[X \neq Y \mid A = a, B = b]\,. \tag{1}$$

If I_N is small this implies that the outcomes of adjacent inputs are almost perfectly correlated – a fact that can be used for secret key agreement.

(a) *Assuming that the boxes allow for a hidden variable model s.t. X and Y can be seen as independent random variables, show that $I_N \geq 1$.*
Hint: *Define X_a to be Alice's outcome when she inputs a and Y_b to be Bob's outcome when he inputs b and consider the quantity*

$$F_N = 1 - \delta_{X_0 Y_{2N-1}} + \sum_{|a-b|=1} \delta_{X_a Y_b}\,, \tag{2}$$

δ_{xy} being the Kronecker-Delta. Show that for any realisation of the different random variables $F_N \geq 1$ and follow that $I_N \geq 1$.

(b) *Within quantum mechanics, e.g. if the boxes contain quantum spins and A and B are inputs defining the measurement basis, one can show that $I_N < 1$ is possible. To see this, assume that Alice and Bob share the 2-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and perform their measurement in the basis $\{|\frac{k\pi}{2N}\rangle, |\frac{k\pi}{2N} + \pi\rangle\}$ for $k \in \{0,1,2,\dots,2N-1\}$ (for Alice $k \in \mathcal{A}$, for Bob $k \in \mathcal{B}$). Here, $|\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$.*
Show that in this case

$$I_N = 2N \sin^2 \frac{\pi}{4N} \leq \frac{\pi^2}{8N}\,. \tag{3}$$

(c) *Consider the case $N = 2$ and compare the above quantum violation of $I_2 \geq 1$ with the violation of the standard Bell inequality.*

Solution.

(a) We consider all possible combinations for the $2N$ measurement outcomes and try to chose them such that $F_N < 1$. The sum in (2) must be zero in this case because otherwise $F_N < 1$ is no longer achievable. Hence, every Kronecker-Delta in this sum must be zero. First, choose $X_0 = 1$. From the following table it follows that in this case $Y_{2N-1} = 0$ and thus $F_N = 1$ because $X_0 \neq Y_{2N-1}$:

X_a	a	b	Y_b
1	0		
		1	0
1	2		
		3	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$2N-2$		
		$2N-1$	0

Analogously we can argue if $X_0 = 0$, thus always $F_N \geq 1$. Since I_N is the expectation value of F_N, the claim follows immediately.

(b) Let us calculate the quantity I_N in the described setting:

$$\begin{aligned}
&P[X = Y \,|\, A = 0, B = 2N-1] \\
&= P[X = Y = 0 \,|\, A = 0, B = 2N-1] + P[X = Y = 1 \,|\, A = 0, B = 2N-1] \\
&= \Big| \underbrace{\langle 0|}_{a=0,x=0} \otimes \underbrace{\langle \tfrac{(2N-1)\pi}{2N}|}_{b=2N-1,y=0} \tfrac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \Big|^2 + \Big| \underbrace{\langle \pi|}_{a=0,x=1} \otimes \underbrace{\langle \tfrac{(2N-1)\pi}{2N} + \pi|}_{b=2N-1,y=1} \tfrac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \Big|^2 \\
&= \frac{1}{2} \Big| \Big(\cos\big((1 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) \langle 00| + \sin\big((1 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) \langle 01| \Big) \Big(|00\rangle + |11\rangle \Big) \Big|^2 \\
&\quad + \frac{1}{2} \Big| \Big(\cos\big((2 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) \langle 10| + \sin\big((2 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) \langle 11| \Big) \Big(|00\rangle + |11\rangle \Big) \Big|^2 \\
&= \frac{1}{2} \cos^2\big((1 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) + \frac{1}{2} \sin^2\big((2 - \tfrac{1}{2N})\tfrac{\pi}{2}\big) \\
&= \frac{1}{2} \Big(\sin^2 \tfrac{\pi}{4N} + \sin^2 \tfrac{\pi}{4N} \Big) = \sin^2 \frac{\pi}{4N} \,,
\end{aligned}$$

where we used the identities $\sin x = \cos(\frac{\pi}{2} - x) = \sin(\pi - x)$ in the second last step.

Likewise we find for $|a - b| = 1$:

$$\begin{aligned}
&P[X \neq Y \,|\, A = 0, B = 2N-1] \\
&= P[X = 0, Y = 1 \,|\, A = a, B = b] + P[X = 1, Y = 0 \,|\, A = a, B = b] \\
&= \Big| \langle \tfrac{a\pi}{2N}| \otimes \langle \tfrac{b\pi}{2N} + \pi| \tfrac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \Big|^2 + \Big| \langle \tfrac{a\pi}{2N} + \pi| \otimes \langle \tfrac{b\pi}{2N}| \tfrac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \Big|^2 \\
&= \frac{1}{2} \big| \cos\big(\tfrac{a\pi}{4N}\big) \cos\big(\tfrac{b\pi}{4N} + \tfrac{\pi}{2}\big) + \sin\big(\tfrac{a\pi}{4N}\big) \sin\big(\tfrac{b\pi}{4N} + \tfrac{\pi}{2}\big) \big|^2 \\
&\quad + \frac{1}{2} \big| \cos\big(\tfrac{a\pi}{4N} + \tfrac{\pi}{2}\big) \cos\big(\tfrac{b\pi}{4N}\big) + \sin\big(\tfrac{a\pi}{4N} + \tfrac{\pi}{2}\big) \sin\big(\tfrac{b\pi}{4N}\big) \big|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\underbrace{-\cos\tfrac{a\pi}{4N}\sin\tfrac{b\pi}{4N}+\sin\tfrac{a\pi}{4N}\cos\tfrac{b\pi}{4N}}_{=\sin\left((a-b)\frac{\pi}{4N}\right)}\Big)^2 + \frac{1}{2}\Big(\underbrace{-\sin\tfrac{a\pi}{4N}\cos\tfrac{b\pi}{4N}+\cos\tfrac{a\pi}{4N}\sin\tfrac{b\pi}{4N}}_{=\sin\left((b-a)\frac{\pi}{4N}\right)}\Big)^2 \\
&= \sin^2\frac{\pi}{4N}\,.
\end{aligned}
$$

Again we used identities for sin and cos, namely $\cos x = \sin(x+\frac{\pi}{2})$ and $\sin x = -\cos(x+\frac{\pi}{2})$. Altogether we find

$$
I_N = \big[1 + (2N-1)\big]\sin^2\frac{\pi}{4N} = 2N\sin^2\frac{\pi}{4N} \le \frac{\pi^2}{8N}\,, \tag{S.1}
$$

because $\sin x \le x$ for $x \ge 0$.

(c) For $N = 2$ we obtain $I_2 = 4\sin^2\frac{\pi}{8} = 2-\sqrt{2} < 1$. The relative violation of the bound $I_2 \ge 1$ is therefore $1 - I_N = \sqrt{2}-1$.

In the standard Bell inequality we have the classical bound to be 2, while quantum mechanics achieves the Tsirelson bound $2\sqrt{2}$. Also here, the relative violation is given by $\frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$. In fact, the standard Bell violation and the violation of I_N for $N = 2$ can be directly connected to each other and are essentially one and the same thing.