Exercise 1. ***Canonical purifications***

Given a state (density operator) ρ on $\mathcal{H}_A$, consider the state $|\psi\rangle_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, defined as

$$|\psi\rangle_{AB} = (\sqrt{\rho_A} \otimes V_{A'\to B})\,|\Omega\rangle_{AA'}, \qquad |\Omega\rangle_{AA'} = \sum_k |k\rangle_A \otimes |k\rangle_{A'}, \tag{1}$$

where $\mathcal{H}_{A'} \simeq \mathcal{H}_A$, $\dim(\mathcal{H}_B) \geq \dim(\mathcal{H}_A)$, and $V_{A'\to B}$ is an isometry from A' to B (i.e. $V^\dagger V = \mathbb{1}_{A'}$).

(a) Show that $|\psi\rangle_{AB}$ is a purification of ρ_A.

(b) Show that every purification of ρ can be written in this form for some $V_{A'\to B}$.

Solution.

(a) Tracing out B, we obtain

$$\begin{aligned}
\mathrm{tr}_B[|\psi\rangle\langle\psi|_{AB}] &= \sqrt{\rho}\,\mathrm{tr}_B[V_{A'\to B}|\Omega\rangle\langle\Omega|_{AA'}V^\dagger_{A'\to B}]\,\sqrt{\rho} \\
&= \sqrt{\rho}\sum_{k,k'}|k\rangle\langle k'|_A\,\mathrm{tr}[V_{A'\to B}|k\rangle\langle k'|_{A'}V^\dagger_{A'\to B}]\,\sqrt{\rho} \\
&= \sqrt{\rho}\sum_{k,k'}|k\rangle\langle k'|_A\,\mathrm{tr}[|k\rangle\langle k'|_{A'}V^\dagger_{A'\to B}V_{A'\to B}]\,\sqrt{\rho} \\
&= \sqrt{\rho}\sum_{k,k'}|k\rangle\langle k'|_A\,\delta_{kk'}\,\sqrt{\rho} \\
&= \sqrt{\rho}\,\mathbb{1}_A\sqrt{\rho} = \rho.
\end{aligned} \tag{S.1}$$

In the fourth line we used that V is an isometry.

(b) We know that any two purifications of ρ_A are related by isometries on the purifying systems, here B. Since applying another isometry to B gives a state of the same form, all purifications can be brought to this form.

Exercise 2. ***Decompositions of density matrices***

Consider a mixed state ρ with two different pure state decompositions

$$\rho = \sum_{k=1}^{d}\lambda_k|k\rangle\langle k| = \sum_{l=1}^{d}p_l|\phi_l\rangle\langle\phi_l|, \tag{2}$$

the former being the eigendecomposition so that $\{|k\rangle\}$ is an orthonormal basis, and the latter involving arbitrary (normalized) states $|\phi_l\rangle$.

(a) Show that the probability vector $\vec{\lambda}$ majorizes *the probability vector $\vec{p}$, which means that there exists a doubly stochastic matrix T_{jk} such that $\vec{p} = T\vec{\lambda}$. The defining property of doubly stochastic, or bistochastic, matrices is that $\sum_k T_{jk} = \sum_j T_{jk} = 1$.*
Hint: *Observe that for a unitary matrix U_{jk}, $T_{jk} = |U_{jk}|^2$ is doubly stochastic.*

(b) The uniform probability vector $\vec{u} = (\frac{1}{d},\ldots,\frac{1}{d})$ is invariant under the action of an $d \times d$ doubly stochastic matrix. Is there an ensemble decomposition of ρ such that $p_l = \frac{1}{d}$ for all l?
Hint: *Try to show that $\vec{u}$ is majorized by any other probability distribution.*

Solution.

(a) By the Proposition presented in class we have $\sqrt{p_l}|\phi_l\rangle = \sum_k \sqrt{\lambda_k} U_{kl}|k\rangle$ for some unitary matrix U_{kl}. Taking the norm of each expression results in

$$p_l = \sum_k \lambda_k |U_{kl}|^2 \tag{S.2}$$

since $|k\rangle$ is an orthonormal basis. Thus $\vec{\lambda}$ majorizes $\vec{p}$. Note that we cannot turn this argument around to say that $\vec{p}$ majorizes $\vec{\lambda}$ since starting from $\sqrt{\lambda_k}|k\rangle = \sum_l \sqrt{p_l} U^\dagger_{kl}|\phi_l\rangle$ we cannot easily compute the norm of the righthand side because the $|\phi_l\rangle$ are not orthogonal.

(b) $\vec{u}$ is majorized by every other distribution $\vec{p}$ (of length less or equal to d) since we can use the doubly stochastic matrix $T_{jk} = 1/d$ for all j,k to produce $\vec{u} = T\vec{p}$. Therefore, to find a decomposition in which all the weights are identical, we need to find a unitary matrix whose entries all have the same magnitude, namely $1/\sqrt{d}$. One choice that exists in every dimension is the Fourier transform $F_{jk} = \frac{1}{\sqrt{d}}\omega^{jk}$, where $\omega = \exp(2\pi i/d)$. The vectors in the decomposition are therefore

$$|\phi_l\rangle = \sum_k \sqrt{\lambda_k}\omega^{kl}|k\rangle. \tag{S.3}$$

Exercise 3. *Generalized measurement by direct (tensor) product*

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle_B$ and the two systems interact via the unitary U_{AB} as follows:

$$|0\rangle_A|0\rangle_B \to \tfrac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B\right), \tag{3}$$

$$|1\rangle_A|0\rangle_B \to \tfrac{1}{\sqrt{6}}\left(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right). \tag{4}$$

(a) Calculate the measurement operators acting on A corresponding to a measurement on B in the canonical basis $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$.

(b) Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?

(c) Suppose A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_A$. What is the state after a measurement, averaging over the measurement result?

Solution.

(a) Name the output states $|\phi_{00}\rangle_{AB}$ and $|\phi_{10}\rangle_{AB}$, respectively. Although the specification of U is not complete, we have the pieces we need, and we can write $U_{AB} = \sum_{jk} |\phi_{jk}\rangle\langle jk|_{AB}$ for some states $|\phi_{01}\rangle$ and $|\phi_{11}\rangle$. The measurement operators A_k are defined implicitly by

$$U_{AB}|\psi\rangle_A|0\rangle_B = \sum_k A_k \otimes \mathbb{1}_B\, |\psi\rangle_A|k\rangle_B. \tag{S.4}$$

Thus $A_k = {}_B\langle k|U_{AB}|0\rangle_B = \sum_j {}_B\langle k|\phi_{j0}\rangle_{AB}\langle j|_A$, which is an operator on system A, even though it might not look like it at first glance. We then find

$$A_0 = \frac{2}{\sqrt{6}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix}. \tag{S.5}$$

(b) The corresponding POVM elements are given by $E_j = A^\dagger A_j$:

$$E_0 = \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \frac{1}{6}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad E_2 = \frac{1}{6}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}. \tag{S.6}$$

They are each rank one (which can be verified by calculating the determinant). The POVM elements project onto the states $|1\rangle, (\sqrt{3}|0\rangle \pm |1\rangle)/2$.

(c) The averaged post-measurement state is given by $\rho' = \sum_j A_j \rho A^\dagger$. In this case we have $\rho' = \mathrm{diag}(2/3, 1/3)$.