Exercise 1. *Partial trace*

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Let ρ_{AB} be a density matrix on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = \mathrm{tr}_B(\rho_{AB})$ the marginal on $\mathcal{H}_A$.

(a) Show that ρ_A is a valid density operator by proving it is:

(i) Hermitian: $\rho_A = \rho_A^\dagger$.

(ii) Positive: $\rho_A \geq 0$.

(iii) Normalised: $\mathrm{tr}(\rho_A) = 1$.

(b) Calculate the reduced density matrix of system A in the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right), \quad where \quad |ab\rangle = |a\rangle_A \otimes |b\rangle_B. \tag{1}$$

(c) Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.

(i) Calculate the marginal distribution P_X for

$$P_{XY}(x,y) = \begin{cases} 0.5 & for\ (x,y) = (0,0), \\ 0.5 & for\ (x,y) = (1,1), \\ 0 & else, \end{cases} \tag{2}$$

with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.

(ii) How can we represent P_{XY} in form of a quantum state?

(iii) Calculate the partial trace of P_{XY} in its quantum representation.

(d) Can you think of an experiment to distinguish the bipartite states of parts (b) and (c)?

Solution.

(a) (i) Remember that ρ_{AB} can always be written as

$$\rho_{AB} = \sum_{i,j,k,l} c_{ij;kl}\ |i\rangle\langle k|_A \otimes |j\rangle\langle l|_B, \tag{S.1}$$

for some bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $c_{ij;kl} = c^\dagger_{kl;ij}$ is hermitian. The reduced density operator ρ_A is then given by

$$\rho_A = \mathrm{tr}_B(\rho_{AB}) = \sum_{i,k} \sum_m c_{im;km} |i\rangle\langle k|_A \tag{S.2}$$

as can easily be verified. Hermiticity of ρ_A follows from

$$\rho_A^\dagger = \sum_{i,k} \sum_m c^\dagger_{im;km} \left(|i\rangle\langle k|_A \right)^\dagger = \sum_{i,k} \sum_m c_{km;im} |k\rangle\langle i|_A = \rho_A. \tag{S.3}$$

(ii) Since $\rho_{AB} \geq 0$ is positive, its scalar product with any pure state is positive. Let $|\psi\rangle_A$ an arbitrary pure state in $\mathcal{H}_A$ and define $|\Psi_m\rangle_{AB} = |\psi\rangle_A \otimes |m\rangle_B$, a state on $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$
\begin{aligned}
0 &\leq \sum_m \langle \Psi_m | \rho_{AB} | \Psi_m \rangle \\
&= \sum_m \langle \psi|_A \otimes \langle m|_B \rho_{AB} |\psi\rangle_A \otimes |m\rangle_B \\
&= \sum_m \sum_{i,j,k,l} c_{ij;kl} \langle \psi|i\rangle\langle k|\psi\rangle_A \langle m|j\rangle\langle l|m\rangle_B \\
&= \sum_{i,k} \sum_m c_{im;km} \langle \psi|i\rangle\langle k|\psi\rangle_A \\
&= \langle \psi|\rho_A|\psi\rangle
\end{aligned} \tag{S.4}
$$

Because this is true for any $|\psi\rangle$ on $mathcal H_A$, it follows that ρ_A is positive.

(iii) Consider

$$
\begin{aligned}
\operatorname{tr}(\rho_A) &= \sum_{i,j} \sum_{m,n} c_{im;km} \langle n|i\rangle\langle k|n\rangle \\
&= \sum_{m,n} c_{nm;nm} = \operatorname{tr}(\rho_{AB}) = 1.
\end{aligned} \tag{S.5}
$$

(b) The reduced state is mixed, even though $|\Psi\rangle$ is pure:

$$
\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big) \tag{S.6}
$$

$$
\operatorname{tr}_B(\rho_{AB}) = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) = \frac{\mathbb{1}_A}{2}. \tag{S.7}
$$

(c) (i) Using $P_X(\cdot) = \sum_{y\in\mathcal{Y}} P_{XY}(\cdot, y)$, we immediately obtain

$$
P_X(0) = 0.5, \quad P_X(1) = 0.5. \tag{S.8}
$$

(ii) A probability distribution $P_Z = \{P_Z(z)\}_z$ may be represented by a state

$$
\rho_Z = \sum_z P_Z(z)|z\rangle\langle z| \tag{S.9}
$$

for a basis $\{|z\rangle\}_z$ of a Hilbert space $\mathcal{H}_Z$. In this case we can create a two-qubit system with composed Hilbert space $\mathcal{H}_X \otimes \mathcal{H}_Y$ in state

$$
\rho_{XY} = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big). \tag{S.10}
$$

(iii) The reduced state of qubit X is

$$
\rho_X = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) = \frac{\mathbb{1}_X}{2}. \tag{S.11}
$$

Notice that the reduced states of this classical state and the Bell state are the same whereas the state of the global state is very different – in particular, the latter is a pure state that can be very useful in quantum communication and cryptography whereas the former is not.

(d) One could for instance measure the two states in the Bell basis,

$$\begin{aligned} |\psi_1\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\ |\psi_3\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \end{aligned} \tag{S.12}$$

The Bell state we analysed corresponds to the first state of this basis, $|\Psi\rangle = |\psi_1\rangle$, and a measurement in the Bell basis would always have the same outcome. For the classical state, however, $\rho_{XY} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)$, so with probability $\frac{1}{2}$ a measurement in this basis will output $|\psi_2\rangle$, and we will know we had the classical state. Of course, if we only have access to a single copy we will find out about the difference only with probability $\frac{1}{2}$. However, with arbitrarily many copies we will find out which state we have with very high probability after a few measurements.

Exercise 2. *Bipartite systems and measurement*

(a) Consider a state ρ_{AB} in a composed system $\mathcal{H}_A \otimes \mathcal{H}_B$ shared by Alice, who is in possession of system A, and Bob, who has access to B. Suppose Alice wants to perform a measurement described by an observable O_A on subsystem $\mathcal{H}_A$. The operator O_A has eigenvalues (possible outcomes) $\{x\}_x$ and may be written as the spectral decomposition $O_A = \sum_x x\, P_x$, where $\{P_x\}_x$ are projectors – operators that only have eigenvalues 0 and 1.

Show that the measurement statistics (probabilities of obtaining the different outcomes) are the same whether you apply $O_A \otimes \mathbb{1}_B$ on the joint state ρ_{AB} or first trace out the system $\mathcal{H}_B$ and then apply O_A on the reduced state ρ_A.

(b) Suppose now that Alice and Bob share a two-qubit system in a maximally entangled state,

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right). \tag{3}$$

Alice then performs a measurement in the basis $\{|\theta\rangle, |\frac{\pi}{2} + \theta\rangle\}$, with $|\theta\rangle := \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$ for $\theta \in \mathbb{R}$, on her qubit. In the notation from (a) the basis corresponds to the projectors $P_{+1} = |\theta\rangle\langle\theta|$ and $P_{-1} = |\frac{\pi}{2} + \theta\rangle\langle\frac{\pi}{2} + \theta|$.

(i) What description does Alice give to system B, given the outcome of her measurement?

(ii) If Bob performs the measurement in the basis $\{|0\rangle, |1\rangle\}$ on his part of the system, B, what is the probability distribution for his outcomes? How would Alice, who knows the outsome of her measurement, describe his probability distribution?

(c) Finally, Alice and Bob share an arbitrary pure state $|\Psi\rangle_{AB}$, and Bob would like to perform a measurement on B described by projectors $\{Q_y\}_y$. Unfortunately his measurement apparatus is broken, however he can still perform arbitrary unitary operations. Meanwhile, Alice's measurement apparatus is in good working order. Show that there exist projectors $\{P_y\}_y$ on Alice's part and unitaries U_y on A and V_y on B so that

$$|\widetilde{\Psi}_y\rangle_{AB} := (\mathbb{1}_A \otimes Q_y)\,|\Psi\rangle_{AB} = (U_y \otimes V_y)\,(P_y \otimes \mathbb{1}_B)\,|\Psi\rangle_{AB}. \tag{4}$$

(Note that the 'state' $|\widetilde{\Psi}_y\rangle_{AB}$ is unnormalized, so that it implicitly encodes the probability of outcome y.)

By showing this we have proven that Alice can assist Bob by performing a related measurement herself, after which they can locally *correct the state using the* local *unitaries U_y and V_y. Notice that Alice will have to (classically) communicate to Bob what her outcome was.*

Hint*: Use the Schmidt decomposition and work in the Schmidt basis of $|\Psi\rangle_{AB}$.*

Solution.

(a) The probability of obtaining outcome x when applying the measurement described by O_A on the reduced state ρ_A is

$$\operatorname{tr}(P_x \rho_A) = \operatorname{tr}\Big(P_x[\operatorname{tr}_B(\rho_{AB})]\Big).$$

On the other hand, the probability of obtaining the same outcome when we apply $O_A \otimes \mathbb{1}_B$ to the global state is

$$\operatorname{tr}([P_x \otimes \mathbb{1}_B]\rho_{AB}) = \operatorname{tr}\Big(\operatorname{tr}_B([P_x \otimes \mathbb{1}_B]\ \rho_{AB})\Big) \tag{S.13}$$

$$= \operatorname{tr}\Big(P_x[\operatorname{tr}_B(\rho_{AB})]\Big), \tag{S.14}$$

where Eq. (S.13) stands because the trace can be decomposed into partial traces and Eq. (S.14) because the partial trace commutes with multiplication with operators of the form $T_A \otimes \mathbb{1}_B$ (see definition of partial trace, e.g. in script).

(b) (i) In accordance with the postulates, Alice describes B by the postmeasurement state. Computing this we find, for any θ,

$${}_A\langle\theta|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}(\cos\theta\,\langle 0| + \sin\theta\,\langle 1|)(|00\rangle + |11\rangle) \tag{S.15}$$

$$= \frac{1}{\sqrt{2}}(\cos\theta\,|0\rangle + \sin\theta\,\langle 1|) = \frac{1}{\sqrt{2}}|\theta\rangle_B. \tag{S.16}$$

Thus, she describes Bob's state as $|\theta\rangle$ or $|\frac{\pi}{2}+\theta\rangle$ depending on the result of her measurement, either of which is equally-likely as the other.

(ii) The calculation from the previous part shows that the outcomes of Alice's measurement on $|\Phi\rangle_{AB}$ are both equally-likely, no matter the value of θ. Hence, also the probability distribution of Bob's outcomes should be uniform, as Alice's mere implementation of the measurement should not affect any observable quantity at his end.

However, we must check this is consistent with the postulates. Conditioned on Alice's measurement result, the state of B is either $|\theta\rangle$ or $|\frac{\pi}{2}+\theta\rangle$, but Bob does not know which, so he must average over the two possibilities. Alice, who knows the result, does not need to average. The probability of obtaining $|0\rangle$ in his measurement, given the state $|\theta\rangle$, is simply $\cos^2\theta$ (this is Alice's decription). bob's 'averaged' probability is then

$$\frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2(\frac{\pi}{2}+\theta) = \frac{1}{2}, \tag{S.17}$$

which is what we expected. We see that Alice and Bob will have different descriptions of the postmeasurement state, which is not surprising as it was assumed that only Alice knows the outcome of her measurement.

(c) Start with the Schmidt decomposition of $|\Psi\rangle_{AB}$, $|\Psi\rangle_{AB} = \sum_k \sqrt{p_k}|\alpha_k\rangle_A|\beta_k\rangle_B$. Bob's measurement projectors Q_y can be expanded in his Schmidt basis as $Q_y = \sum_{kl} c^y_{kl}|\beta_k\rangle\langle\beta_l|$. In order for Alice's measurement to replicate Bob's, the probabilities of the various outcomes must be identical, which is to say

$$\begin{aligned} &\langle\Psi|\mathbb{1}_A \otimes Q_y|\Psi\rangle_{AB} = \langle\Psi|P_y \otimes \mathbb{1}_B|\Psi\rangle_{AB} \\ \Longrightarrow\quad &\sum_k p_k\langle\alpha_k|P_y|\alpha_k\rangle = \sum_k p_k\langle\beta_k|Q_y|\beta_k\rangle \end{aligned} \tag{S.18}$$

for all y. Thus Alice should choose $P_y = \sum_{kl} c^y_{kl} |\alpha_k\rangle\langle\alpha_l|$. The post-measurement states when Alice or Bob measures are given by

$$|\Psi'_y\rangle_{AB} = \sum_{kl} \sqrt{p_k} c^y_{kl} |\alpha_l\rangle_A |\beta_k\rangle_B \qquad \text{and} \qquad |\Psi_y\rangle_{AB} = \sum_{kl} \sqrt{p_k} c^y_{kl} |\alpha_k\rangle_A |\beta_l\rangle_B, \tag{S.19}$$

respectively. Neither is in Schmidt form, but note that they are related by a simple swap operation $|\alpha_j\rangle_A|\beta_k\rangle_B \leftrightarrow |\alpha_k\rangle_A|\beta_j\rangle_B$, which is unitary; call this unitary W_{AB} so that $|\Psi_y\rangle = WYket\Psi_y$. Now let $U'_y \otimes V'_y$ be unitary operators which transform $|\Psi_y\rangle$ to Schmidt form in the $|\alpha_j\rangle|\beta_k\rangle$ basis. That is, $(U'_y \otimes V'_y)|\Psi_y\rangle = \sum_k \sqrt{p^y_k}|\alpha_k\rangle|\beta_k\rangle$, and it follows that $W(U'_y \otimes V'_y)|\Psi_y\rangle = (U'_y \otimes V'_y)|\Psi_y\rangle$. Therefore $V'_y \otimes U'_y$ takes $|\Psi'_y\rangle$ to Schmidt form:

$$(V'_y \otimes U'_y)|\Psi'_y\rangle = WW^\dagger(V'_y \otimes U'_y)W|\Psi_y\rangle = W(U'_y \otimes V'_y)|\Psi_y\rangle = \sum_k \sqrt{p^y_k}|\alpha_k\rangle|\beta_k\rangle, \tag{S.20}$$

and thus

$$\begin{aligned} & (U'_y \otimes V'_y)|\Psi_y\rangle = (V'_y \otimes U'_y)|\Psi'_y\rangle \\ \Rightarrow\quad & (U'_y \otimes V'_y)(\mathbb{1}_A \otimes Q_y)|\Psi\rangle = (V'_y \otimes U'_y)(P_y \otimes \mathbb{1}_B)|\Psi\rangle \\ \Rightarrow\quad & (\mathbb{1}_A \otimes Q_y)|\Psi\rangle = (U'^\dagger_y V'_y \otimes V'^\dagger_y U'_y)(P_y \otimes \mathbb{1}_B)|\Psi\rangle. \end{aligned} \tag{S.21}$$

In other words, $P_y = \sum_{kl} c^y_{kl}|\alpha_k\rangle\langle\alpha_l|$, $U_y = U'^\dagger_y V'_y$ and $V_y = V'^\dagger_y U'_y$ do the job.