**Exercise 1.** ***Canonical purifications***

Given a state (density operator) $\rho$ on $\mathcal{H}_A$, consider the state $|\psi\rangle_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, defined as

$$|\psi\rangle_{AB} = (\sqrt{\rho_A} \otimes V_{A'\to B})\,|\Omega\rangle_{AA'}, \qquad |\Omega\rangle_{AA'} = \sum_k |k\rangle_A \otimes |k\rangle_{A'}, \tag{1}$$

where $\mathcal{H}_{A'} \simeq \mathcal{H}_A$, $\dim(\mathcal{H}_B) \geq \dim(\mathcal{H}_A)$, and $V_{A'\to B}$ is an isometry from $A'$ to $B$ (i.e. $V^\dagger V = \mathbb{1}_{A'}$).

(a) Show that $|\psi\rangle_{AB}$ is a purification of $\rho_A$.

(b) Show that every purification of $\rho$ can be written in this form for some $V_{A'\to B}$.

**Exercise 2.** ***Decompositions of density matrices***

Consider a mixed state $\rho$ with two different pure state decompositions

$$\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k| = \sum_{l=1}^{d} p_l |\phi_l\rangle\langle\phi_l|, \tag{2}$$

the former being the eigendecomposition so that $\{|k\rangle\}$ is an orthonormal basis, and the latter involving arbitrary (normalized) states $|\phi_l\rangle$.

(a) Show that the probability vector $\vec{\lambda}$ *majorizes* the probability vector $\vec{p}$, which means that there exists a doubly stochastic matrix $T_{jk}$ such that $\vec{p} = T\vec{\lambda}$. The defining property of doubly stochastic, or bistochastic, matrices is that $\sum_k T_{jk} = \sum_j T_{jk} = 1$.
*Hint:* Observe that for a unitary matrix $U_{jk}$, $T_{jk} = |U_{jk}|^2$ is doubly stochastic.

(b) The uniform probability vector $\vec{u} = (\frac{1}{d}, \ldots, \frac{1}{d})$ is invariant under the action of an $d \times d$ doubly stochastic matrix. Is there an ensemble decomposition of $\rho$ such that $p_l = \frac{1}{d}$ for all $l$?
*Hint:* Try to show that $\vec{u}$ is majorized by any other probability distribution.

**Exercise 3.** ***Generalized measurement by direct (tensor) product***

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, $A$, by first coupling it to a three-level system, $B$, and then making a projective measurement on the latter. $B$ is initially prepared in the state $|0\rangle_B$ and the two systems interact via the unitary $U_{AB}$ as follows:

$$|0\rangle_A|0\rangle_B \to \tfrac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B\right), \tag{3}$$

$$|1\rangle_A|0\rangle_B \to \tfrac{1}{\sqrt{6}}\left(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right). \tag{4}$$

(a) Calculate the measurement operators acting on $A$ corresponding to a measurement on $B$ in the canonical basis $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$.

(b) Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?

(c) Suppose $A$ is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_A$. What is the state after a measurement, averaging over the measurement result?