Exercise 1. ***Trace distance***

The trace distance (or L_1-distance) between two probability distributions P_X and Q_X over a discrete alphabet $\mathcal{X}$ is defined as

$$\delta(P_X, Q_X) = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| . \tag{1}$$

The trace distance may also be written as

$$\delta(P_X, Q_X) = \max_{\mathcal{S} \subseteq \mathcal{X}} |P_X[\mathcal{S}] - Q_X[\mathcal{S}]| , \tag{2}$$

where we maximise over all events $S \subseteq \mathcal{X}$ and the probability of an event $\mathcal{S}$ is given by $P_X[\mathcal{S}] = \sum_{x \in \mathcal{S}} P_X(x)$.

(a) Show that $\delta(\cdot, \cdot)$ is a good measure of distance by proving that $0 \leq \delta(P_X, Q_X) \leq 1$ and the triangle inequality $\delta(P_X, R_X) \leq \delta(P_X, Q_X) + \delta(Q_X, R_X)$ for arbitrary probability distributions P_X, Q_X and R_X.

(b) Show that definitions (2) and (1) are equivalent.

(c) Let us now find an operational meaning for the trace distance. Suppose that P_X and Q_X represent the probability distributions of the outcomes of two dice that look identical. You are allowed to throw one of them only once and then have to guess which die that was. What is your best strategy? What is the probability that you guess correctly and how can you relate that to the trace distance $\delta(P_X, Q_X)$?

Exercise 2. ***Weak law of large numbers***

Let A be a positive random variable with expectation value $\langle A \rangle = \sum_a a \, P_A(a)$. Let $P[A \geq \varepsilon]$ denote the probability of an event $\{A \geq \varepsilon\}$ for some $\varepsilon > 0$.

(a) Prove Markov's inequality

$$P[A \geq \varepsilon] \leq \frac{\langle A \rangle}{\varepsilon}. \tag{3}$$

(b) Use Markov's inequality to prove Chebyshev's inequality, i.e.,

$$P\left[(X - \mu)^2 \geq \epsilon\right] \leq \frac{\sigma^2}{\epsilon}, \tag{4}$$

where σ denotes the standard deviation of X.

(c) Use Chebyshev's inequality to prove the weak law of large numbers for i.i.d. X_i:

$$\lim_{n \to \infty} P\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)^2 \geq \varepsilon\right] = 0 \quad \text{for any } \varepsilon > 0, \mu = \langle X_i \rangle . \tag{5}$$

Exercise 3. ***Conditional probabilities I: Mrs. Smith's children***

(a) You are strolling in a park when a woman, Mrs. Smith, is approaching you with a covered twin buggy. She tells you that she has fraternal twins and in the course of the conversation you learn that one of the twins is a girl named Jane. What is the probability that the other twin is a girl, too?

(b) We now change the situation slightly. After having a short conversation with Mrs. Smith you know that she has fraternal twins. You ask her 'are they both boys?' and she answers 'no'. What is the probability that she has two girls?

(c) Another version of this story goes as follows: during the conversation with Mrs. Smith you ask 'could you please tell me the sex of one of your twins?' and she answers 'one of them is a girl'. What is now the probability that the other is a girl?

(d) Explain the difference between the three situations in words and in terms of conditional probabilities. What are the hidden assumptions?

Exercise 4. ***Conditional probabilities II: how knowing more does not always help***

Suppose you are visiting your grandfather in his hut in Scotland. You had offered him a radio for Christmas three years ago, but he is not so fond of such modern technologies and has not used it since. You decide to initiate a game to prove to him that technology is helpful: every evening you alone listen to the weather forecast on the radio and then both you and your grandfather try to guess if it will rain next morning. Having lived there since birth, your grandfather knows that it rains on 80% of the days. You had reached the same conclusion on previous summer holidays. You also know that the weather forecast is right 80% of the time and is always correct when it predicts rain.

(a) What is the optimal strategy for your grandfather? And for you?

(b) Both of you keep a record of your guesses and the actual weather for statistical analysis. After some months who will have guessed correctly more often?

(c) Can you think of an argument to convince him that listening to the forecast is useful?