Exercise 1. ***Around the neutron star***

A satellite is moving around a neutron star (whose outside geometry is described by the Schwarzschild metric) at the distance $r = 7M$ on a circular orbit.

1. What is the orbital period as measured by the observer at infinity?
2. What is the orbital period measured by the satellite itself?

Exercise 2. ***Perturbation of a circular orbit***

Show that the perturbation of an unstable circular orbit of a massive particle grows exponentially with the proper time:

$$r(\tau) = r_{circ} + \delta r_0 e^{K\tau} + O((\delta r)^2). \tag{1}$$

Determine K.

Exercise 3. ***Escape the black hole***

In the exterior Schwarzschild geometry, a particle is thrown within the plane $\theta = \pi/2$ from a radial position R with initial proper angular velocity $\omega = d\phi/d\tau$ such that its initial velocity vector is orthogonal to the radial direction. Find the minimum value of ω such that the particle reaches infinity.

Exercise 4. ***Closed Timelike Curves***

In this exercise, we want to repeat Gödel's proof of the existence of solutions to Einstein's equation that allow for timelike closed curves.

i) Let a be a constant. Consider the line element

$$ds^2 = a^2\left(dx_0^2 - dx_1^2 + \frac{1}{2}e^{2x_1}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0dx_2\right) . \tag{2}$$

Find the metric components of g_{ab} and show that, for a specific choice of a and λ, this metric satisfies Einstein's equation with positive cosmological constant [1],

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi\rho u_i u_j + \lambda g_{ij} , \tag{3}$$

where u^i is the unit vector in the direction of x_0. Determine the relations between ρ, a and λ.

[1] The prefactor (here λ) to an additional term proportional to the metric in Einstein's equation is commonly called cosmological constant.

We now switch to cylindrical coordinates (t, r, ϕ) on each subspace with $x_3 \equiv \text{const}$ and define $x_3 = 2y$. It can be shown that the line element in these coordinates takes the form

$$ds^2 = 4a^2 \left(dt^2 - dr^2 - dy^2 + \left(\sinh^4 r - \sinh^2 r \right) d\phi^2 + 2\sqrt{2} \sinh^2 r d\phi dt \right) . \tag{4}$$

ii) Consider the spherical curve $\mathcal{C}$ given by

$$\mathcal{C} : t = y = 0 \,, \quad r = R \,, \quad \phi \in [0, 2\pi[\tag{5}$$

and find the set of values of R such that $\mathcal{C}$ is timelike.

iii) Now let us define the curve

$$\mathcal{C}_\alpha : y = 0 \,, \quad r = R \,, \quad t = -\alpha\phi \,, \quad \phi \in [0, 2\pi[\,, \tag{6}$$

and argue that for sufficently small ϵ and $|\alpha| < \epsilon$ the curve $\mathcal{C}_\alpha$ is also timelike.

Let P and Q be points on the t-line and let P precede Q. With the result above, show that there exists a timelike curve that starts at Q and ends at P.

We have just proven that within Einstein's Theory of General Relativity, there exist solutions that allow for closed timelike curves. In particular, it is possible in these worlds to travel back in time or influence the past from the future.