Exercise 1. ***Electrodynamics***

The general covariant form of Maxwell equations is

$$\nabla^a F_{ab} = -4\pi j_b \ , \qquad \nabla_{[a} F_{bc]} = 0 \ . \tag{1}$$

Show that these equations imply charge conservation

$$\nabla_a j^a = 0 \tag{2}$$

i) by making use of the antisymmetry $F_{ab} = -F_{ba}$;

ii) by writing F_{ab} in terms of a potential A_b,

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \ , \tag{3}$$

and imposing the covariant Lorentz gauge condition $\nabla^a A_a = 0$.

Exercise 2. ***Pressureless perfect fluid (dust)***

The stress-energy tensor of a pressureless perfect fluid (dust) is

$$T^{\mu\nu} = \rho\, u^\mu u^\nu \ , \tag{4}$$

where ρ is the energy density and u^μ the four-velocity. The equations of motion of this model of matter describe the motion of the fluid. However, in the case of dust they also describe the motion of the particles themselves, since the distances between the particles are constant. Show that the covariant conservation of the stress-energy tensor

$$\nabla_a T^{ab} = 0 \tag{5}$$

implies that the particles are moving along geodesics.

Hint: Use $u_\nu u^\nu = c^2$.

Exercise 3. ***Einstein's equation and electrodynamics***

We denote by $T^{\mu\nu}$ the stress-energy tensor of electrodynamics.

i) Let A_μ be the potential (in n dimensions), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F_{\sigma\rho} g^{\mu\sigma} g^{\nu\rho} \tag{6}$$

the Lagrangian in the absence of sources. Show that

$$T^{\mu\nu} = F^\mu{}_\alpha F^{\alpha\nu} - \frac{1}{4}(F_{\alpha\beta} F^{\beta\alpha}) g^{\mu\nu} \ . \tag{7}$$

Hint: Use

$$\delta_g \int_D \mathcal{L}\sqrt{-g}\,\mathrm{d}^n x = -\frac{1}{2}\int_D T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\,\mathrm{d}^n x \ , \tag{8}$$

where $D \subset M$ is a compact region in space-time and δ_g denotes the variation with respect to the metric. Furthermore, use (and prove)

$$(\delta g^{\mu\nu})\, g_{\nu\sigma} = -g^{\mu\nu}(\delta g_{\nu\mu}) \ , \quad \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\delta g_{\beta\alpha} \ . \tag{9}$$

ii) Show that in four dimensions ($n = 4$), Einstein's equations $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ with $T_{\mu\nu}$ the stress-energy tensor of electrodynamics, imply $R = 0$.

iii) Derive the corresponding expression for R in general dimension n.