Exercise 1. ***Inverse metric***

Prove that

$$g^{\nu\sigma}\partial_\mu g_{\nu\sigma} = \frac{1}{g}\frac{\partial g}{\partial x^\mu} \ , \tag{1}$$

where $g = det(g_{\mu\nu})$ is the determinant of the spacetime metric $g_{\mu\nu}$, which depends on the coordinates x^ρ.

Hint: You may want to prove first that $\mathrm{Tr}\ \log M = \log\ \det M$ for a symmetric square matrix M. In order to do so, you may want to use that any symmetric square matrix M can be diagonalised.

Exercise 2. ***Curvature tensor***

A metric $g_{\mu\nu}$ can be described in terms of a line element as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Consider the metric defined by

$$ds^2 = e^{2\Omega(t,x)}\left(-dt^2 + dx^2\right) \ , \tag{2}$$

where $\Omega(t,x)$ is a scalar function. Calculate the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ for this spacetime in components as well as using the tetrad formalism introduced in the lecture.

Exercise 3. ***Killing vector fields***

In Exercise sheet 4 we learned that

$$(L_X Y)^b = X^a\nabla_a Y^b - Y^a\nabla_a X^b \ . \tag{3}$$

(a) Since the Lie derivative has the Leibnitz property and commutes with contractions, deduce from this result the action of the Lie derivative on 1-forms

$$(L_X\omega)_b = X^a\nabla_a\omega_b + \omega_a\nabla_b X^a \tag{4}$$

and on 2-forms

$$(L_X T)_{ab} = X^c\nabla_c T_{ab} + T_{cb}\nabla_a X^c + T_{ac}\nabla_b X^c \ . \tag{5}$$

Hint: By definition, $L_X f = Xf = X^a\nabla_a f$.

(b) Show that the formulae (4) and (5) are in fact independent of the choice of covariant derivative.

(c) Suppose $\phi_t : M \to M$ is a one-parameter group of isometries, $\phi_t^* g = g$, where g is the metric on M. Show that the generating vector field X that is defined by $X(f) = \frac{d}{dt}(f\circ\phi_t)\big|_{t=0}$ satisfies the Killing vector equation

$$\nabla_a X_b + \nabla_b X_a = 0 \ , \tag{6}$$

where ∇_a is the covariant derivative with respect to which the metric is covariantly constant.

(d) For the 4-dimensional Minkowski spacetime write down the Killing vectors generating boosts. Show that they satisfy the Killing equation (6).