Exercise 1. ***Differential forms***

Check that the exterior derivative of a 2-form Ω, defined by

$$\begin{aligned} \mathrm{d}\Omega(X_1, X_2, X_3) =& X_1(\Omega(X_2, X_3)) - X_2(\Omega(X_1, X_3)) + X_3(\Omega(X_1, X_2)) \\ & - \Omega([X_1, X_2], X_3) + \Omega([X_1, X_3], X_2) - \Omega([X_2, X_3], X_1) \end{aligned} \tag{1}$$

defines indeed a 3-form, i.e., that

$$\mathrm{d}\Omega(fX_1, X_2, X_3) = \mathrm{d}\Omega(X_1, fX_2, X_3) = \mathrm{d}\Omega(X_1, X_2, fX_3) = f\mathrm{d}\Omega(X_1, X_2, X_3)\ . \tag{2}$$

Exercise 2. ***Lie derivative***

In components, the action of the Lie derivative L_X on a vector field R is given as

$$(L_X(R))^\mu = \frac{\partial R^\mu}{\partial x^\nu} X^\nu - R^\nu \frac{\partial X^\mu}{\partial x^\nu}\ , \tag{3}$$

while the action on a 1-form ω is

$$(L_X(\omega))_\mu = \frac{\partial \omega_\mu}{\partial x^\nu} X^\nu + \omega_\nu \frac{\partial X^\nu}{\partial x^\mu}\ . \tag{4}$$

i) Show that

$$L_X(Y) = [X, Y]\ . \tag{5}$$

ii) Check that

$$L_{[X,Y]} = L_X L_Y - L_Y L_X \tag{6}$$

when applying both sides to vector fields and 1-forms.

Exercise 3. ***Integration of forms***

Let us consider an n-dimensional orientable manifold M, i.e., a manifold for which all transition functions satisfy that

$$\det\left(\frac{\partial x'}{\partial x}\right) > 0\ . \tag{7}$$

i) Suppose ω is an n-form, whose support is contained in a single chart. Using the arguments from Exercise 3 ii) of Sheet 3, we can write the n-form as

$$\begin{aligned} \omega \;=\; \frac{1}{n!}\omega_{i_1\cdots i_n}(x)\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \;=\; \tilde{\omega}(x)\, \frac{1}{n!}\varepsilon_{i_1\cdots i_n} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\ = \tilde{\omega}(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n\ , \end{aligned} \tag{8}$$

where $\varepsilon_{i_1\cdots i_n}$ is the Levi-Civita **symbol** and $\tilde{\omega}(x)$ is a **scalar density**.

Show that the integral

$$\int_M \omega := \int_M \tilde{\omega}(x)\, \mathrm{d}x^1 \cdot \mathrm{d}x^2 \cdot \cdots \cdot \mathrm{d}x^n \tag{9}$$

is well-defined, i.e., independent of the specific coordinates that are being used (we only consider charts for which the support of ω is contained within).

ii) Now suppose that the n-dimensional manifold M has a boundary ∂M, and that ω is an $(n-1)$-form that has support in one chart (which may now also intersect the boundary). Prove Stokes' Theorem

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \ . \tag{10}$$

[Hint: See Fig. 1 and consider separately the two depicted cases: a) (left) the chart in which ω has support contains the boundary and the boundary satisfies $x^1 = 0$, b) (right) the chart in which ω has support does not contain the boundary.]

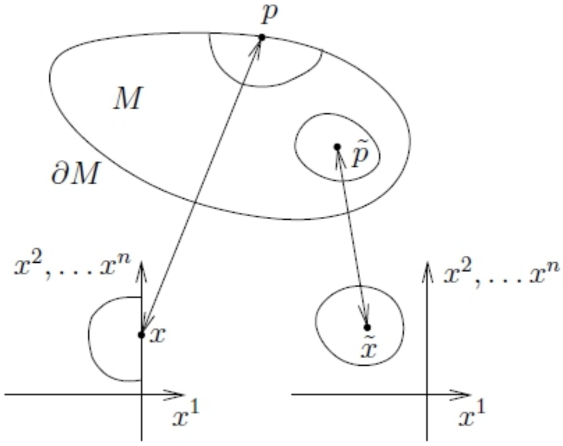

Figure 1: An n-dimensional manifold M with $(n-1)$-dimensional boundary ∂M is locally homeomorph to $\mathbb{R}^n_- = \{(x^1, \dots, x^n) \in \mathbb{R}^n | x^1 \leq 0\}$ and we can always define local coordinates $(x^1, \dots, x^n)$ for M, in which ∂M satisfies $x^1 = 0$.

Exercise 4. ***Inner product***

Let X be a vector field and Ω a p-form. We define the $(p-1)$-form $(i_X \Omega)$ by

$$(i_X \Omega)(Y_1, \dots, Y_{p-1}) = \Omega(X, Y_1, \dots, Y_{p-1})$$

Prove the following properties:

i) Let Ω_i be a p_i-form for $i = 1, 2$. Then

$$i_X(\Omega_1 \wedge \Omega_2) = i_X(\Omega_1) \wedge \Omega_2 + (-1)^{p_1} \Omega_1 \wedge i_X(\Omega_2) \ .$$

ii) We have the identity $i_X^2 = 0$.

iii) If f is a function then

$$i_X(\mathrm{d}f) = (\mathrm{d}f)(X) = X(f) \ .$$

iv) The Lie derivative L_X, acting on 0 and 1-forms, can be written as

$$L_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X \ ,$$

where d is the exterior derivative.

Exercise 5. ***Electrodynamics in form language***

In this problem we consider the familiar theory of electromagnetism on a 4-dimensional manifold with signature $(-,+,+,+)$. Maxwell's theory involves a one-form electromagnetic potential $A = A_\mu \, \mathrm{d}x^\mu$ from which one constructs the field-strength (a two-form) via the exterior derivative,

$$F = \mathrm{d}A = \frac{1}{2!} F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{11}$$

i) Show that the components of F are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using that the exterior derivative is nilpotent, $\mathrm{d}^2 = 0$, deduce that $\mathrm{d}F = 0$. Write $\mathrm{d}F = 0$ in components — how many independent equations are there? — and relate them to the homogeneous Maxwell equations.

[Hint: Deduce from the definition of the 4-potentials that the field strength tensor is given (in flat space) as

$$F_{\mu\nu} \sim \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} .]$$

ii) Define the Hodge dual $\star F$ of the field strength 2-form F via

$$\star \, F = \frac{1}{2!} \frac{F_{\mu\nu}}{2!} \epsilon^{\mu\nu}{}_{\rho\sigma} \, \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ , \tag{12}$$

where $\epsilon_{\mu\nu\rho\sigma} = \sqrt{|g|} \varepsilon_{\mu\nu\rho\sigma}$ (Levi-Civita tensor) and $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. [The Hodge dual can be defined more generally for p-forms in D dimensions, and it is in general a $D-p$ form; for the case of a 2-form in $D=4$ dimensions, the Hodge dual is therefore again a 2-form.] Show that

$$F \wedge \star F = \alpha \sqrt{|g|} F_{\mu\nu} F^{\mu\nu} \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{13}$$

and

$$F \wedge F = \star F \wedge \star F = \beta \sqrt{|g|} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ , \tag{14}$$

and find the explicit value of the numeric constants α and β. Both of these quantities are 4-forms and may be integrated over the spacetime manifold. The combination (13) is proportional to the Lagrangian density that yields Maxwell's equations in the absence of sources as equations of motion. Use Stokes' theorem to show that the integral of (14) reduces to a boundary term (on a manifold with boundary).

[Hint: you may want to use that the Levi-Civita symbol in D dimensions satisfies the identity

$$\varepsilon^{\mu_1 \ldots \mu_r \mu_{r+1} \ldots \mu_D} \varepsilon_{\nu_1 \ldots \nu_r \mu_{r+1} \ldots \mu_D} = (D-r)! \, \delta^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_r} \ , \tag{15}$$

where $\delta^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_r} = r! \delta^{\mu_1}{}_{[\nu_1} \delta^{\mu_2}{}_{\nu_2} \cdots \delta^{\mu_r}{}_{\nu_r]}$ is the antisymmetrised product of r Kronecker deltas (notice that the $r!$ factor in front cancels, for example $\delta^{\mu_1\mu_2}_{\nu_1\nu_2} = \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}$, etc). This can for example be used to write:

$$\mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_n} = \frac{1}{n!(D-n)!} \varepsilon^{\mu_1 \cdots \mu_{D-n} \nu_1 \cdots \nu_n} \varepsilon_{\mu_1 \cdots \mu_{D-n} \cdots \alpha_1 \cdots \alpha_n} \, \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_n} \ .] \tag{16}$$

iii) The adjoint exterior derivative is defined via the identity

$$\mathrm{d}^\dagger F = \star\, \mathrm{d}(\star F) \ ,$$

where the Hodge dual of a 3-form ω in 4 dimensions equals

$$(\star\omega)_{\mu_1} \ = \ \frac{1}{3!}\omega_{\nu_1\,\nu_2\,\nu_3}\epsilon^{\nu_1\nu_2\nu_3}{}_{\mu_1} \ , \tag{17}$$

and $\epsilon_{\nu_1\nu_2\nu_3\mu_1}$ is again the Levi-Civita tensor.

Assemble the electric charge density ρ and the electric vector current density $\vec{J}$ into a one-form current $J = \eta_{\mu\nu} J^\nu dx^\mu = -\rho\, dt + \vec{J} \cdot d\vec{x}$. Show that the inhomogeneous Maxwell equations (in flat space)

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{18}$$

can be written succinctly in form language as

$$\mathrm{d}^\dagger F = J \ . \tag{19}$$

[Note that the operator $\mathrm{d}^\dagger$ is in fact the adjoint of d with respect to the inner product defined on p-forms by the integral

$$(\omega, \eta) = \int_M \omega \wedge \star\eta \ ,$$

and $\star$ is the Hodge dual that maps a p-form to a $D - p$ form. In particular we have $(\mathrm{d}\chi, \eta) = (\chi, \mathrm{d}^\dagger\eta)$ if η and χ are a p-form and $(p-1)$-form, respectively.]