Exercise 1. ***Vector fields***

A vector field is a linear map on the space of differentiable functions $\mathcal{F}$

$$X : \mathcal{F} \to \mathcal{F}$$

satisfying the derivation property,

$$X(fg) = X(f)\,g + fX(g)\ .$$

i) Show that $X \circ Y$ and $Y \circ X$ are *not* vector fields, but that the commutator,

$$[X, Y] = X \circ Y - Y \circ X,$$

is a vector field.

ii) Confirm that the commutator also satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ . \tag{1}$$

Exercise 2. ***Decomposition of a*** $(2, 0)$ ***tensor to its symmetric and antisymmetric part***

i) Show that any $(2, 0)$ tensor M can be written as $M = S+A$, where S is a totally symmetric $(2, 0)$ tensor and A is a totally antisymmetric $(2, 0)$ tensor. Write the components of S and A explicitly in terms of those of M.

ii) Show that the $(0, 2)$ tensors constructed from S and A via the metric (i.e. $S_{ij} = g_{ia}g_{jb}S^{ab}$, etc) have the same symmetry properties as their $(2, 0)$ counterparts, and that

$$S_{ab}A^{ab} = S^{ab}A_{ab} = 0\ .$$

iii) Let T be an arbitrary $(2, 0)$ tensor. Show that $S_{ab}T^{ab} = S_{ab}T^{(ab)}$ and $A_{ab}T^{ab} = A_{ab}T^{[ab]}$, where

$$T^{(ab)} = \frac{1}{2}\big(T^{ab} + T^{ba}\big)\ , \qquad T^{[ab]} = \frac{1}{2}\big(T^{ab} - T^{ba}\big)\ .$$

Exercise 3. ***The Levi-Civita symbol and tensor***

The *Levi-Civita symbol* in n dimensions is defined as

$$\varepsilon_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_n} = \begin{cases} +1 & \text{for even permutations of } 012\ldots n-1 \\ 0 & \text{if } \mu_i = \mu_j \\ -1 & \text{for odd permutations of } 012\ldots n-1 \end{cases} \tag{2}$$

or, equivalently, as the totally antisymmetric symbol with $\varepsilon_{012\ldots n-1} = +1$.

i) Show that if we require that the components of this object take the same value in *every* chart, then they must transform as

$$\varepsilon'_{\mu_1 \ldots \mu_n} = \det\left(\frac{\partial x'}{\partial x}\right) \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}} \varepsilon_{\nu_1 \ldots \nu_n} \ . \tag{3}$$

An object with this property is called a *tensor density of weight 1* — the weight refers to the power of the determinant of the Jacobian in the transformation property.

[Hint: It is useful to notice that the determinant of a general matrix Λ (here expanded along the columns) can be written as

$$\det(\Lambda) = \Lambda^{\nu_1}{}_0 \Lambda^{\nu_2}{}_1 \ldots \Lambda^{\nu_n}{}_{n-1} \varepsilon_{\nu_1 \ldots \nu_n} \ . \tag{4}$$

This can be taken as the definition of the determinant, or proven by induction if so desired.]

ii) Let V be an n-form (i.e., a totally antisymmetric $(0, n)$-tensor) on an n-dimensional space. Show that this object has only one independent component, and that it can therefore be written as

$$V_{\mu_1 \ldots \mu_n} = \varepsilon_{\mu_1 \ldots \mu_n} \phi(x) \ . \tag{5}$$

How should $\phi(x)$ transform in order to respect the tensorial character of V?

iii) Use the results from the previous two parts to show that, under a coordinate transformation $x^\mu \to x'^\mu$, the components of the above n-form satisfy

$$V'_{\mu_1 \ldots \mu_n} = \det\left(\frac{\partial x}{\partial x'}\right) V_{\mu_1 \ldots \mu_n} \ . \tag{6}$$

iv) Now assume that the manifold M is endowed with a metric g, and that the transition functions between any two overlapping charts of the atlas satisfy

$$\det\left(\frac{\partial x'}{\partial x}\right) > 0 \ . \tag{7}$$

On such manifold we may define the *Levi-Civita tensor* ϵ by

$$\epsilon = \sqrt{|g(x)|}\, \varepsilon \ , \tag{8}$$

where $g(x)$ is the determinant of the metric g at point x. Prove that ϵ is a tensor. [The requirement of eq. (7) on the atlas implies that M is an orientable manifold, for which this choice of charts then defines an orientation.]