12.1. Veneziano amplitude

The four-point open-string tachyon scattering amplitude $A(k_1, k_2, k_3, k_4)$ (usually known as the *Veneziano amplitude*) is given by the correlation function of four tachyon vertex operators inserted on the boundary of a disk. Each partial amplitude is then reduced to a single integral of the form

$$A(k_1, k_2, k_3, k_4) \;=\; g_s\, \delta^{(26)}\left(\sum_{i=1}^{4} k_i\right) \int_0^1 \mathrm{d}x\, |x|^{2\alpha' k_1\cdot k_2} |1-x|^{2\alpha' k_2\cdot k_3}\,. \qquad (12.1)$$

Recall that the Euler gamma function is defined as

$$\Gamma(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\, e^{-t}\, t^x \qquad \text{for} \quad \mathrm{Re}(x) > 0\,. \qquad (12.2)$$

a) Show that the integral in eq. (12.1) evaluates to

$$A(k_1, k_2, k_3, k_4) \;=\; g_s\, \delta^{(26)}\left(\sum_{i=1}^{4} k_i\right) \frac{\Gamma(-\alpha' s - 1)\,\Gamma(-\alpha' t - 1)}{\Gamma[-\alpha'(s+t) - 2]}\,, \qquad (12.3)$$

where

$$s = -(k_1 + k_2)^2\,, \qquad t = -(k_2 + k_3)^2 \qquad k_i^2 = \frac{1}{\alpha'}\,, \qquad (12.4)$$

the latter being the mass-shell condition for tachyons.
Hint: show that

$$\Gamma(a+b) \int_0^1 \mathrm{d}y\, y^{a-1} (1-y)^{b-1} \;=\; \Gamma(a)\,\Gamma(b)\,. \qquad (12.5)$$

for $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$.

b) The gamma function has simple poles at negative integers $z = 0, -1, -2, -3, \ldots$, and $\frac{1}{\Gamma(z)}$ is an entire function.
Consider the poles of the amplitude. What do these poles correspond to?

12.2. T-duality for open strings

We have seen that T-duality establishes the equivalence between the closed string spectra obtained via compactification over two circles of radii R and $\tilde{R} = \alpha'/R$, respectively. This identification requires the exchange of momentum and winding modes of the compactified strings.

For open strings, the situation is rather different. Open strings compactified along x^{25} have no winding number w^{25}, therefore it would naïvely seem that there is no notion of T-duality. However, it is possible to show that T-duality along a direction x^p relates the spectrum of open strings in the presence of a Dp-brane wrapped around S^1 to the spectrum of open strings in the presence of a D$(p-1)$-brane localized in the compact direction x^p. Schematically:

$$\text{T-duality along } x^p \;:\; [\,\mathrm{D}p, R\,] \Longleftrightarrow [\,\mathrm{D}(p-1),\, \tilde{R} = \alpha'/R\,]\,. \qquad (12.6)$$

If this relation is true, it is evident that T-duality along x^p must change the boundary conditions along that direction! We will focus now on $p = 25$, but the generalization to arbitrary D-branes is straightforward.

Consider a spacetime with a space-filling D25-brane and compactified x^{25}. Focus on the coordinated field $X^{25}(\tau, \sigma) \equiv X(\tau, \sigma)$ of open strings; it obeys NN boundary conditions.

a) Write down the mode expansion for $X(\tau, \sigma)$. What is the difference with the ordinary, non-compactified case? *Hint:* think about the zero-modes.

b) We have seen that in the case of T-duality for closed strings, the T-duality transformation is equivalent to flipping the sign of the right-movers. Perform the same transformation in the case of open strings, and show that

$$\tilde{X}(\tau, \sigma) \equiv X_L(\tau, \sigma) - X_R(\tau, \sigma) \tag{12.7}$$

satisfies DD boundary conditions.

c) Show that the open string described by $\tilde{X}$ stretches an interval $2\pi n\alpha'/R$ along x^{25}, with $n \in \mathbb{Z}$. Since $\tilde{X}$ satisfies DD boundary conditions, it means it has no momentum along the x^{25} direction. What kind of quantum number is n then?

Argue why this configuration is equivalent to a single D24-brane localized at some fixed position on a circle of radius $\tilde{R} = \alpha'/R$.

d) Show that the spectrum of the open string Hamiltonian is indeed invariant under the above T-duality transformation.