11.1. Orientifolds and D-branes

In this exercise we consider a stack of N D25-branes. In particular, we study a specific truncation of the open string spectrum, obtained by keeping the states invariant under the reversal of orientation of the open string. The operator Ω that flips the orientation acts on the transverse directions X^I, $I = 2, \ldots 25$ as

$$\Omega\, X^I(\tau,\sigma)\,\Omega^{-1} \;=\; X^I(\tau, \pi-\sigma)\,. \tag{11.1}$$

We moreover require that $\Omega\, x_0^-\,\Omega^{-1} = x_0^-$ and $\Omega\, p^+\,\Omega^{-1} = p^+$.

a) Derive the action of Ω on oscillators. What is the expected action of Ω on α_n^-? Does this expectation hold?

b) Consider first the case $N = 1$. Assuming the ground states $|p^+, \vec{p}\rangle$ to be Ω-invariant, compute the eigenvalue of Ω on the spectrum; work out explicitly the states that survive the projection with $N^\perp \le 2$. What happens to the massless states? Can you explain why?

c) Consider now a stack of N $D25$-branes. Explain why the assumption

$$\Omega|p^+, \vec{p}; [ij]\rangle = |p^+, \vec{p}; [ji]\rangle\,, \tag{11.2}$$

where $[ij]$ labels the two D-branes on which the string begins and ends, is reasonable. Enumerate the ground states of these open strings. *Hint:* a generic linear combination of the non-orientifolded ground states is $\sum_{i,j} \lambda_{ij}|p^+, \vec{p}; [ij]\rangle$; find the constraint on the λ_{ij}'s that make the states invariant and count the number of ground states of the orientifolded theory.

d) We now want to work out the full open string spectrum of the theory. Consider the states

$$\sum_{ij} \lambda_{ij}|\Psi; [ij]\rangle\,, \tag{11.3}$$

where Ψ labels the open-string state and i, j are the labels of the starting and ending $D25$-brane, respectively. What are the conditions that $\Omega|\Psi; [ij]\rangle$ and λ_{ij} must obey for a state to belong to the theory? How many states and what kind of states are left at the massless level? What happens to the U(N) gauge symmetry of the non-orientifolded model?

11.2. String stretched between Dp and Dq branes

Consider a pair of parallel Dp and Dq branes with $p > q$; by "parallel" in this case we mean that the directions are split as

$$\underbrace{x^0, x^1, \ldots, x^q,}_{\{x^\pm,\, x^i\}\text{ common tangential coordinates}} \qquad \underbrace{x^{q+1}, x^{q+2}, \ldots, x^p,}_{\{x^r\}\text{ mixed coordinates}} \qquad \underbrace{x^{p+1}, x^{p+2}, \ldots, x^{D-1},}_{\{x^a\}\text{ common normal coordinates}} \tag{11.4}$$

where $i = 2, \ldots q$, $\quad r = q+1, \ldots, p \quad$ and $a = p+1, \ldots, D-1$. The position of the Dp brane is given by the coordinates x_0^a and the position of the Dq brane by y_0^r and y_0^a.

Consider the open string stretching from the Dp to the Dq brane.

a) What boundary conditions must the coordinate fields $X^r(\tau, \sigma)$ satisfy?

b) Compute the mode expansion for X^r. *Hint:* as usual, split the coordinate fields as

$$X^r(\tau, \sigma) = f^r(\tau + \sigma) + g^r(\tau - \sigma) . \tag{11.5}$$

Impose the correct boundary conditions derived before. Define then the modes α_j^r in such a way that the combination $\dot{X}^r \pm X'^r$ reads

$$[\dot{X}^r \pm X'^r](\tau, \sigma) \sim \sum_j \alpha_j^r \, e^{-ij(\tau \pm \sigma)} . \tag{11.6}$$

Pay particular attention to the allowed values for the index j!
Compute the commutation relation for the α_j^r modes.

c) [*Advanced*] Compute the M^2 operator. *Hint:* start from the equality (you should be able to derive it yourself!)

$$2\alpha' p^+ p^- = \alpha' p^i p^i + \frac{1}{2}\alpha_0^a \alpha_0^a + \sum_{n=1}^{\infty} \left[\alpha_{-n}^i \alpha_n^i + \alpha_{-n}^a \alpha_n^a \right] + \frac{1}{2} \sum_j \alpha_{-j}^r \alpha_j^r + a \, , \tag{11.7}$$

where we have restored the normal-ordering constant a (and where j runs over the values determined in the previous steps). Recall what is the eigenvalue of α_0^a for DD coordinates. The last sum must be normal-ordered. Schematically, you should get a sum resembling the following

$$\sum_j \alpha_{-j}^r \alpha_j^r = 2 \sum_{j>0} \alpha_{-j}^r \alpha_j^r + \sum_{j>0} [\alpha_j^r, \alpha_{-j}^r] . \tag{11.8}$$

Regularise the last sum using ζ-function regularisation (again, be careful about the allowed values for j). Show then that

$$M^2 = \left[\frac{x_0^a - y_0^a}{2\pi\alpha'} \right]^2 + \frac{1}{\alpha'} \left[N^\perp - 1 + \frac{1}{16}(p - q) \right] . \tag{11.9}$$

Consider now a string stretching from the Dq-brane to the Dp-brane.

d) Write the boundary conditions satisfied by the X^r coordinates. Use them to derive the mode expansion along the lines of the previous computation. What are the hermiticity properties of the oscillators?

e) Compute $\dot{X}^r \pm X'^r$ and compare with the result you got for eq. (11.S23). How does M^2 change?

f) Start from the solution to the string stretched from the Dp-brane to the Dq-brane and reverse its orientation by sending $\sigma \mapsto \pi - \sigma$. Compare it with the solution you got for the string stretched from Dq to Dp; compare in particular the mode expansions and explain why the hermiticity conditions you found are consistent.