7.1. Physical states and upper bound on critical dimension

We define the generators $K_m = k_\mu \alpha^\mu_m$ with $k^\mu = \frac{1}{2}(1, 0 \ldots, 0, -1)$, and denote by $\tilde{L}_{-2}$ the combination $\tilde{L}_{-2} = L_{-2} + \frac{3}{2}L_{-1}^2$, where L_m are the Virasoro generators with $c = D$.

a) Show that the commutation relations between the L_m and the K_n are of the form

$$[L_m, K_n] = -nK_{m+n} \ , \qquad [K_m, K_n] = 0 \ . \tag{7.1}$$

b) Consider the state

$$|\psi\rangle = \left[\tilde{L}_{-2} + \left(\frac{D}{2} - 13\right)(K_{-2} + 2K_{-1}^2)\right]|l\rangle \ , \tag{7.2}$$

where $|l\rangle$ has momentum squared $l^2 = -1$ so that $L_0|l\rangle = -|l\rangle$ (we work with $\alpha' = 1$) and moreover $K_m|l\rangle = 0$ for $m > 0$. Show that $|\psi\rangle$ is physical for any D. Calculate its norm and show that

$$\langle\psi|\psi\rangle = 13 - \frac{D}{2} \ . \tag{7.3}$$

In particular, this calculation shows that $D = 26$ is the upper critical dimension. *Hint:* compute $L_2|\psi\rangle$, $L_1|\psi\rangle$ and $K_0|l\rangle$.

7.2. Basis for Fock space

With the notations of the previous question, let $\mathcal{F}^M$ be the space of states $|f\rangle$ that satisfy

$$K_m|f\rangle = L_m|f\rangle = 0 \qquad \text{for } m > 0\,, \tag{7.4}$$

as well as $R|f\rangle = M|f\rangle$, where $R = L_0 - p^2$ counts the excitation number. We take the momentum to be of the form $p = p_0 + Lk$ (where L is an arbitrary integer) with $p_0 = (0, 0, \ldots, 0, 1)$. We introduce the short-hand notation

$$|\{\lambda, \mu\}, f, M, \nu\rangle = L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-n}^{\mu_n} |f, M, \nu\rangle \ , \tag{7.5}$$

where $|f, M, \nu\rangle$ is an orthonormal basis for $\mathcal{F}^M$ — the different basis vectors are labelled by ν.

We make the following claim: the vectors of the form

$$|\{\lambda, \mu\}, f, M, \nu\rangle \tag{7.6}$$

give a basis for the states with $R = M + \sum_r r\lambda_r + \sum_s s\mu_s = N$, and, as N varies, for the whole vector space of states with momentum of the form $p = p_0 + Lk$. This means that at a given level N, the *full* space of states can be decomposed into $\mathcal{F}^N$, as well as the states that descend from $\mathcal{F}^{N'}$, $N' < N$ through eq. (7.5).

a) Verify this claim for $R = 1$ and a given $|p\rangle$. *Hint:* Consider all descendants generated by α^{μ}_{-m}. Do the analysis for general spacetime dimension D.

b) Under the assumption that the claim holds, count the number of states at $R = 2$ that satisfy eq. (7.4). What do you expect for $R > 2$? (Both of these questions should be answered for general D.)

c) Verify the claim for $R = 2$, $D = 4$ and $p = (0, 0, 0, 1)$. *Hint:* compute the descendants from the states at $R = 0, 1$ via the action of L_{-i}, K_{-i}, $i > 0$. Count then the states that are annihilated by L_1, L_2, K_1, K_2. Confirm that your answer agrees with b) above.