4.1. Consistency of solutions in light-cone gauge

We have seen during the lecture that a convenient choice of gauge is the so-called light-cone gauge, where we set[2]

$$X^+ = \alpha' p^+ \tau \,. \tag{4.1}$$

With this choice of gauge, one can show that the coordinate X^- is fully determined by the equation

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2p^+\alpha'}\left(\dot{X}^I \pm X^{I\prime}\right)^2 . \tag{4.2}$$

a) Use eq. (4.1) to determine $\partial_\tau X^-$ and $\partial_\sigma X^-$. Moreover, show that the consistency condition $\partial_\tau(\partial_\sigma X^-) = \partial_\sigma(\partial_\tau X^-)$ holds if the transverse coordinates X^I satisfy the wave equations.

b) Show that X^- as calculated before satisfies the wave equation if the transverse coordinates X^I satisfy the wave equation.

c) Show that, given an arbitrary assignment of Dirichlet/Neumann boundary conditions for the transverse coordinates X^I, the coordinate X^- will always satisfy Neumann boundary conditions.

4.2. Light-cone gauge and mode expansion

The mode expansion of a solution to the equations of motion can be expressed in general as

$$X^\mu(\sigma,\tau) = x_0^\mu + \kappa^2 p^\mu \tau + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \,; \tag{4.3}$$

here $\kappa^2 = 2\alpha'$ for closed strings and α' for open strings.

The Noether currents associated with the Poincaré symmetries of the Polyakov action (in conformal gauge) are

$$\mathcal{P}_\mu^\alpha = -T\partial^\alpha X_\mu \,, \qquad \mathcal{J}_{\mu\nu}^\alpha = P_\mu^\alpha X_\nu - P_\nu^\alpha X_\mu \,, \tag{4.4}$$

where μ,ν are spacetime indices and α,β are worldsheet indices.

a) Express $\mathcal{P}_0^\mu$ in terms of the mode expansion. Calculate the Lorentz charge

$$J^{\mu\nu} = \int_0^{2\pi} \mathrm{d}\sigma\, \mathcal{J}_0^{\mu\nu} \,, \tag{4.5}$$

and check that it is conserved. *Hint:* use

$$\int_0^{2\pi} \mathrm{d}\sigma\, e^{\pm in\sigma} = 2\pi\delta_{n,0} \tag{4.6}$$

[2] This is the expression for open strings, the equivalent for closed strings has an additional factor of 2.

and the fact that the commutator $[\alpha_n^\mu, \tilde{\alpha}_n^\nu] = 0$ even for the quantized theory. Be careful with signs!

b) Express J^{-I} in terms of the mode expansion for $J^{\mu\nu}$ derived above. In a quantum field theory, symmetry generators should be realised by hermitian operators

$$(J^{\mu\nu})^\dagger = J^{\mu\nu} \,. \tag{4.7}$$

Assuming canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, show that the naïve construction of J^{-I} is *not* hermitian. Rewrite the Lorentz charge J^{-I} such that it is a hermitian operator.

4.3. The Polyakov action

Consider the Polyakov action for relativistic strings

$$S_P = -\frac{T}{2} \int \mathrm{d}^2\sigma \, \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,. \tag{4.8}$$

a) Compute the equations of motion by varying S_P with respect to the metric $h^{\alpha\beta}$ and to the component fields X^μ. *Hint:* the variation of the determinant of the metric is $\delta h = -h\, h_{\alpha\beta} \delta h^{\alpha\beta}$. Moreover, recall that the energy-momentum tensor can be defined as the response of the system to an infinitesimal variation of the metric, $\delta S := -T \int \mathrm{d}^2\sigma \sqrt{h} T_{\alpha\beta} \delta h^{\alpha\beta}$.

b) Show that $S_P = S_{NG}$ when the equations of motion for $h_{\alpha\beta}$ are satisfied (*Hint:* compute $\det[\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}]$). Is it true also off-shell? Explain why.

c) Show that S_P is invariant under a reparametrisation $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$.

d) Show that S_P is invariant under Weyl transformations, i.e. local angle-preserving transformations of the metric $h_{\alpha\beta} \to e^{2\omega(\sigma)} h_{\alpha\beta}$. Show that this so-called Weyl symmetry implies the tracelessness of the energy-momentum tensor. *Hint:* for the last point, consider an infinitesimal Weyl transformation

$$\delta h_{\alpha\beta} = 2\omega\, h_{\alpha\beta} \,, \qquad \delta X^\mu = 0 \,. \tag{4.9}$$

e) *[Advanced]* Consider the Poincaré transformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu \,, \tag{4.10}$$

for constant Λ and a. Use the Noether procedure to show that in conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ the corresponding conserved currents are

$$\mathcal{P}_\mu^\alpha = -T\, \partial^\alpha X_\mu \,, \qquad \mathcal{J}_{\mu\nu}^\alpha = \mathcal{P}_\mu^\alpha X_\nu - \mathcal{P}_\nu^\alpha X_\mu \,. \tag{4.11}$$