2.1. Open string endpoints motion

Consider the motion of a relativistic string with free endpoints. Use the explicit form of P^σ_μ to compute $P^\sigma_\mu P^{\sigma\,\mu}$, and show that the appropriate boundary conditions imply that the endpoints move at the speed of light. *Hint:* exploit the reparametrization invariance of the problem to choose a suitable gauge.

[For future reference: notice that the fact that the endpoints move at the speed of light is a straightforward consequence of the Virasoro constraints for the Polyakov action.]

2.2. Relativistic point particle

The action of a relativistic point particle is proportional to the length of its worldline, that is

$$S_{\rm rp} = -\,\alpha \int_\gamma {\rm d}s\,, \tag{2.1}$$

with the line element

$${\rm d}s^2 = -\eta_{\mu\nu}\,{\rm d}X^\mu {\rm d}X^\nu = c^2{\rm d}t^2 - {\rm d}\vec{x}^2\,, \tag{2.2}$$

and α a constant to be determined. γ is the path between two points X^μ_1 and X^μ_2, parametrized by a parameter τ. The action is then

$$S_{\rm rp} = -\,\alpha \int_\gamma {\rm d}\tau \sqrt{-\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau}}\,. \tag{2.3}$$

a) What could the constant α be? Determine its value by first parametrizing the path by the time coordinate t and then taking the nonrelativistic limit $|\vec{v}| \ll c$; identify the first terms in the expansion. What happens when the mass of the particle vanishes?

b) Derive the equations of motion by varying the action in eq. (2.3) (you may set $c = 1$ from now on). *Hint:* compute the canonically conjugate momentum P_μ, then explain the result.

c) Show that the form of the action is invariant under reparametrizations $\tau' = \tau'(\tau)$.

d) Suppose the relativistic particle is electrically charged with charge q. The coupling of the particle with an external electromagnetic potential A_μ is governed by the action

$$S_{e.m.} = \frac{q}{c}\int {\rm d}\tau\, A_\mu(X)\,\frac{\partial X^\mu}{\partial\tau}\,. \tag{2.4}$$

Consider the action $S = S_{\rm rp} + S_{\rm e.m.}$ and find the equations of motion for the particle. *Hint:* compute the variation of the action under the variation of the path δX^μ (including the *full* variation of the second term); use P_μ from above to simplify the expressions.

e) Consider now the following action. We introduce an auxiliary field, an "einbein" e for the worldline metric and write the action as

$$S_{\rm p} = \int {\rm d}\tau \left[e^{-1}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau} - m^2 e\right]. \tag{2.5}$$

e.1) Derive the equations of motion by varying the action w.r.t X and e.

e.2) Show that the action S_{p} is equivalent to S_{rp} by solving for the einbein and substituting the solution back into the action. What is the advantage of S_{p} with respect to S_{rp}?

e.3) Consider an infinitesimal reparametrization $\delta\tau = -\epsilon(\tau)$. Show that the action is invariant under such transformation. *Hint:* the variation of the einbein is (check it!)

$$\delta e = \frac{\partial}{\partial \tau}[e\,\epsilon(\tau)]\,. \tag{2.6}$$