Exercise 1. *Magnetic domain wall.*

We want to calculate the energy of a magnetic domain wall in the framework of the Ginzburg-Landau (GL) theory. Assuming translational symmetry in the (y, z)-plane, the GL functional in zero field reads

$$F[m, m'] = F_0 + \int dx \left\{ \frac{A}{2} m(x)^2 + \frac{B}{4} m(x)^4 + \frac{\kappa}{2} [m'(x)]^2 \right\}. \tag{1}$$

(a) Solve the GL equation with boundary conditions

$$m(x \to \pm\infty) = \pm m_0, \quad m'(x \to \pm\infty) = 0, \tag{2}$$

where m_0 is the magnetization of the uniform solution.

Solution. The Euler-Lagrange equation of the GL functional is

$$0 = \frac{\delta F}{\delta m} = \frac{\partial f}{\partial m} - \frac{d}{dx}\frac{\partial f}{\partial m'} = Am + Bm^3 - \kappa m''. \tag{S.1}$$

Assuming $A < 0$ and $B > 0$ the uniform solution is

$$m_0 := \sqrt{-\frac{A}{B}}. \tag{S.2}$$

By introducing rescaled variables $s = x/\xi$ and $u(s) = m(s\xi)/m_0$, where

$$\xi = \sqrt{-\frac{\kappa}{A}}$$

is the correlation length, we arrive at the equation

$$u(s) - u(s)^3 + u''(s) = 0. \tag{S.3}$$

Multiplying the above equation by u' and integrating from $-\infty$ to s we obtain

$$u'(s)^2 = \frac{1}{2}\left[1 - u(s)^2\right]^2$$

where we have used $u(-\infty) = -1$ and $u'(-\infty) = 0$. The correct solution for u' is the positive root,

$$u'(s) = \frac{1}{\sqrt{2}}\left[1 - u(s)^2\right]$$

which can be integrated to give

$$u(s) = \tanh\left[\frac{s - s_0}{\sqrt{2}}\right] \quad \Longrightarrow \quad m(x) = m_0 \tanh\left[\frac{x - x_0}{\sqrt{2}\xi}\right]. \tag{S.4}$$

Without loss of generality we set $x_0 = s_0\xi = 0$ in the following.

(b) First, find the energy of the uniformly polarized solution (no domain walls). Next, compute the energy of the solution with a domain wall compared to the uniform solution. Use the coefficients A, B and κ according to the expansion of the mean-field free energy of the Ising model (see Eqs. (6.80) and (6.85)). Finally, find the energy of a sharp step in the magnetization and compare it to the above results.

Solution. The free energy density of the uniformly polarized solution is $f_u = f_0 + Am_0^2/4$. The energy of the domain wall as compared to the uniform solution is therefore

$$\begin{aligned}
\Delta F &= \int dx \left\{ \frac{A}{2} m(x)^2 + \frac{B}{4} m(x)^4 + \frac{\kappa}{2} [m'(x)]^2 - \frac{A}{4} m_0^2 \right\} \\
&= \int dx \left\{ \frac{m(x)}{2} \left[Am(x) + \frac{B}{2} m(x)^3 - \kappa m''(x) \right] - \frac{A}{4} m_0^2 \right\} \\
&= \int dx \left[-\frac{B}{4} m(x)^4 - \frac{A}{4} m_0^2 \right] \\
&= -\frac{Am_0^2}{4} \int dx \left[1 - \frac{m(x)^4}{m_0^4} \right].
\end{aligned}$$

In the second line we have used integration by parts and in the third line we have used the GL equation. Changing to the integration variable $t = x/(\sqrt{2}\xi)$ yields

$$\Delta F = -\frac{Am_0^2}{4} \sqrt{2}\xi \int dt \left[1 - (\tanh t)^4 \right], \tag{S.5}$$

and by using $\tanh' x = 1 - \tanh^2 x$ we find

$$\begin{aligned}
\Delta F &= -\frac{Am_0^2}{4} \sqrt{2}\xi \int dt \left[1 - \tanh^2 t (1 - \tanh' t) \right] \\
&= -\frac{Am_0^2}{4} \sqrt{2}\xi \int dt \left[(\tanh t)' + \frac{1}{3} (\tanh^3 t)' \right] \\
&= -\frac{2Am_0^2}{3} \sqrt{2}\xi.
\end{aligned} \tag{S.6}$$

Using the expressions of Chapter 6 (see Eqs. (6.80) and (6.85)) for the coefficients A, B and κ (derived for an Ising model with coarse graining), we find that

$$\Delta F \sim Jm_0^2 \sqrt{1 - \frac{T}{T_c}} \to 0 \quad (T \to T_c). \tag{S.7}$$

In contrast, a sharp step in the magnetization from $-m_0$ to m_0 costs an energy

$$E \sim Jm_0^2\,, \tag{S.8}$$

(see Chap. 6.6), which for $T \to T_c$ is less favorable.

[1]

Note that in the above energy discussions, the actual position of the domain wall (see entropy contribution in Chap. 6.6) was not taken into account.

[1]Notice that (S.4) describes a sharp step in the magnetization if $\xi \to 0$. One might think then that (S.8) contradicts the expression (S.6), as the latter goes to zero if $\xi \to 0$ while the former does not. However, one should keep in mind that the continuum model considered here is derived from a discrete model by coarse-graining. In particular, ξ depends on the lattice spacing a and the reduced temperature τ according to

$$\xi \propto \frac{a}{\sqrt{\tau}}\,.$$

The correlation length ξ is thus always greater than a and can not be zero. The continuum limit keeps the information about the discreteness of the original model. A sharp step in the discrete case corresponds to a step of width a. At zero temperature, when the system is freezed and $\xi = a$, both expressions (S.8) and (S.7) agree.

Exercise 2. ***Temperature dependence of the superfluid fraction.***

In the lecture we calculated the number of condensed (superfluid) particles at zero temperature [Eq. (7.31)]. In this exercise we want to determine the temperature dependence of this fraction in the limit $T \to 0$.

(a) Calculate the expectation value of the density of particles with momentum $\boldsymbol{k}$,

$$n_{\boldsymbol{k}} := \frac{1}{\Omega} \left\langle \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \right\rangle . \tag{3}$$

(b) Approximate the temperature dependence of this density,

$$\delta n_{\boldsymbol{k}}(T) := n_{\boldsymbol{k}}(T) - n_{\boldsymbol{k}}(T=0) , \tag{4}$$

in the limit $T \to 0$.

(c) Calculate the temperature dependence of the density of condensed particles,

$$\delta n_0 = -\sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} , \tag{5}$$

in the same limit. What happens in a two-dimensional system?

Hint. Keep only the terms of lowest order in T.

(d) Calculate the expectation value $\left\langle \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{-\boldsymbol{k}} \right\rangle$. What is the physical interpretation of this quantity?

Solution.

(a) The Bose-Einstein distribution for the Bogolyubov quasiparticles reads

$$\left\langle \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}} \right\rangle = \frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} . \tag{S.9}$$

This allows us to easily calculate the particle number

$$\begin{aligned} \Omega n_{\boldsymbol{k}} &= \left\langle \left(u_{\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}} - v_{\boldsymbol{k}} \hat{\gamma}_{-\boldsymbol{k}} \right) \left(u_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}} - v_{\boldsymbol{k}} \hat{\gamma}^\dagger_{-\boldsymbol{k}} \right) \right\rangle \\ &= u_{\boldsymbol{k}}^2 \left\langle \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}} \right\rangle + v_{\boldsymbol{k}}^2 \left\langle \hat{\gamma}_{-\boldsymbol{k}} \hat{\gamma}^\dagger_{-\boldsymbol{k}} \right\rangle - u_{\boldsymbol{k}} v_{\boldsymbol{k}} \left(\langle \hat{\gamma}_{\boldsymbol{k}} \hat{\gamma}_{-\boldsymbol{k}} \rangle + \left\langle \hat{\gamma}^\dagger_{-\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}} \right\rangle \right) \\ &= u_{\boldsymbol{k}}^2 \left\langle \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}} \right\rangle + v_{\boldsymbol{k}}^2 \left\langle \hat{\gamma}^\dagger_{-\boldsymbol{k}} \hat{\gamma}_{-\boldsymbol{k}} + 1 \right\rangle \\ &= \left(u_{\boldsymbol{k}}^2 + v_{\boldsymbol{k}}^2 \right) \frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} + v_{\boldsymbol{k}}^2 \\ &= \frac{1 + \chi_{\boldsymbol{k}}^2}{1 - \chi_{\boldsymbol{k}}^2} \frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} + \frac{\chi_{\boldsymbol{k}}^2}{1 - \chi_{\boldsymbol{k}}^2} . \end{aligned} \tag{S.10}$$

(b) At $T = 0$ the first term vanishes as $\beta \to \infty$ and $E_{\boldsymbol{k}} > 0$, while the second term is independent from temperature. Therefore the density difference is given by

$$\Omega\, \delta n_{\boldsymbol{k}} = \frac{1 + \chi_{\boldsymbol{k}}^2}{1 - \chi_{\boldsymbol{k}}^2} \frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} . \tag{S.11}$$

In the limit $T \to 0$ we find $\beta \to \infty$ such that the exponential $\mathrm{e}^{\beta E_{\boldsymbol{k}}}$ is strongly peaked around $\boldsymbol{k} = 0$. Therefore we can approximate $\chi_{\boldsymbol{k}}$ for $k \to 0$. There we obtain

$$\begin{aligned}\chi_{\boldsymbol{k}} &= 1 + \frac{\hbar^2 k^2}{2mUn_0} - \sqrt{\left(1 + \frac{\hbar^2 k^2}{2mUn_0}\right)^2 - 1} \\ &= 1 + \frac{\hbar^2 k^2}{2mUn_0} - \sqrt{\frac{\hbar^2 k^2}{mUn_0} + \left(\frac{\hbar^2 k^2}{2mUn_0}\right)^2} \\ &\approx 1 - \frac{\hbar k}{\sqrt{mUn_0}} \,.\end{aligned} \tag{S.12}$$

This leads to an approximation of the temperature-independent part of $\delta n_{\boldsymbol{k}}$:

$$\begin{aligned}\frac{1+\chi_{\boldsymbol{k}}^2}{1-\chi_{\boldsymbol{k}}^2} &\approx \frac{1 + 1 - \frac{2\hbar k}{\sqrt{mUn_0}} + \frac{\hbar^2 k^2}{Un_0 m}}{1 - 1 + \frac{2\hbar k}{\sqrt{mUn_0}} - \frac{\hbar^2 k^2}{Un_0 m}} \\ &\approx \frac{2 - \frac{2\hbar k}{\sqrt{mUn_0}}}{\frac{2\hbar k}{\sqrt{mUn_0}}} = \frac{\sqrt{mUn_0}}{\hbar k} - 1\end{aligned} \tag{S.13}$$

For finite $\boldsymbol{k}$, where $\beta E_{\boldsymbol{k}} \gtrsim 1$, we can approximate the Bose-Einstein distribution by the Boltzmann distribution

$$\frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} \approx \mathrm{e}^{-\beta E_{\boldsymbol{k}}} \,, \tag{S.14}$$

where we use the linear approximation for the energy,

$$E_{\boldsymbol{k}} \approx \sqrt{\frac{Un_0}{m}}\,\hbar k =: \hbar k c_{\mathrm{s}} \,. \tag{S.15}$$

Therefore, we obtain the approximation

$$\Omega\, \delta n_{\boldsymbol{k}} \approx \left(\frac{mc_{\mathrm{s}}}{\hbar k} - 1\right) \mathrm{e}^{-\beta \hbar k c_{\mathrm{s}}} \,. \tag{S.16}$$

(c) In three dimensions, the density of condensed particles is given by

$$\delta n_0 = -\sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} = -\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \delta n_{\boldsymbol{k}} \,. \tag{S.17}$$

Here, we insert the approximation (S.16) and obtain

$$\begin{aligned}\Omega\, \delta n_0 &\approx -\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(\frac{mc_{\mathrm{s}}}{\hbar k} - 1\right) \mathrm{e}^{-\beta \hbar k c_{\mathrm{s}}} \\ &= -\frac{1}{2\pi^2} \int \mathrm{d}k \left(\frac{mc_{\mathrm{s}}}{\hbar}\, k - k^2\right) \mathrm{e}^{-\beta \hbar k c_{\mathrm{s}}} \\ &= -\frac{1}{2\pi^2} \left(\frac{mc_{\mathrm{s}}}{\hbar(\beta\hbar c_{\mathrm{s}})^2} - \frac{2}{(\beta \hbar c_{\mathrm{s}})^3}\right) \\ &\overset{T\to 0}{\approx} -\frac{mk_{\mathrm{B}}^2}{2\pi^2 \hbar^3 c_{\mathrm{s}}}\, T^2 \,.\end{aligned} \tag{S.18}$$

In two dimensions, a similar calculation would lead to a linear temperature-dependence,

$$\delta n_0 \propto T \,. \tag{S.19}$$

However, in this calculation we underestimated the contributions for very small k: For $\beta E_{\boldsymbol{k}} \lesssim 1$ we can approximate the Bose-Einstein distribution by

$$\frac{1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}} - 1} \approx \frac{1}{\beta E_{\boldsymbol{k}}} \approx \frac{1}{c_{\mathrm{s}} \hbar k} \,. \tag{S.20}$$

Due to the factor of k in the three-dimensional case, this contribution can be neglected. However, in two dimensions, the integral for δn_0 diverges,

$$\Omega\,\delta n_0 \approx \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{m c_\mathrm{s}}{\hbar k}\frac{1}{c_\mathrm{s}\hbar k} = \frac{m}{2\pi\hbar^2}\int \frac{\mathrm{d}k}{k} \longrightarrow \infty\,, \tag{S.21}$$

such that there exists no superfluid condensate at finite temperature.

(d) We can perform a similar calculation as in Eq. (S.10):

$$\begin{aligned}
\left\langle \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}\right\rangle &= \left\langle \left(u_{\boldsymbol{k}}\hat{\gamma}^\dagger_{\boldsymbol{k}} - v_{\boldsymbol{k}}\hat{\gamma}_{-\boldsymbol{k}}\right)\left(u_{\boldsymbol{k}}\hat{\gamma}^\dagger_{-\boldsymbol{k}} - v_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}}\right)\right\rangle \\
&= -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(\left\langle\hat{\gamma}^\dagger_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}}\right\rangle + \left\langle\hat{\gamma}_{-\boldsymbol{k}}\hat{\gamma}^\dagger_{-\boldsymbol{k}}\right\rangle\right) + u^2_{\boldsymbol{k}}\left\langle\hat{\gamma}^\dagger_{\boldsymbol{k}}\hat{\gamma}^\dagger_{-\boldsymbol{k}}\right\rangle + v^2_{\boldsymbol{k}}\left\langle\hat{\gamma}_{\boldsymbol{k}}\hat{\gamma}_{-\boldsymbol{k}}\right\rangle \\
&= -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(\frac{2}{\mathrm{e}^{\beta E_{\boldsymbol{k}}}-1}+1\right) \\
&= -\frac{\chi_{\boldsymbol{k}}}{1-\chi^2_{\boldsymbol{k}}}\frac{\mathrm{e}^{\beta E_{\boldsymbol{k}}}+1}{\mathrm{e}^{\beta E_{\boldsymbol{k}}}-1} \\
&= -\frac{\chi_{\boldsymbol{k}}}{1-\chi^2_{\boldsymbol{k}}}\left[\tanh\left(\tfrac{1}{2}\beta E_{\boldsymbol{k}}\right)\right]^{-1}\,.
\end{aligned} \tag{S.22}$$

This quantity can be physically understood as the rate at which particles are exchanged with the condensate.