Exercise 1. ***Ising Model: Infinite-Range Forces and Mean Field.***

Consider an Ising model where now *all* spins interact between each other with the same strength $J = 1/N$ (long-range forces). The Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2N}\sum_{i,k} s_i s_k - H\sum s_i \ . \tag{1}$$

The coupling constant is rescaled by N so that the total energy remains finite; also the factor one-half compensates the fact that in the sum, each index i and k ranges independently from 1 to N, and thus we counted each bond twice.

In this exercise, we will show that the mean-field approach for this model is exact (at least for $N \to \infty$).

(a) In order to calculate the partition function for this model, we will introduce a little mathematical trick. Show that the Boltzmann factor which appears in the partition function can be written as

$$\mathrm{e}^{-\beta\mathcal{H}} = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} d\lambda \exp\left(-\frac{N\beta\lambda^2}{2} + \sum_i \beta\,(\lambda + H)\, s_i\right) \ . \tag{2}$$

This is a particular case of the Gaussian transform method which will be seen in the lecture.

Hint. Remember the Gaussian integral $\int dx\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}$ *and complete the square.*

Solution. To show that the Boltzmann term can be written as in Equation (2), one can first complete the square in the exponent (introducing $M = \sum s_i$),

$$\begin{aligned}
-\frac{N\beta\lambda^2}{2} + \sum_i \beta\,(\lambda+H)\,s_i &= -\frac{N\beta\lambda^2}{2} + \beta\lambda M + \beta H M = -\frac{N\beta}{2}\left[\lambda^2 - \frac{2M}{N}\lambda - \frac{2HM}{N}\right] \\
&= -\frac{N\beta}{2}\left[\left(\lambda - \frac{M}{N}\right)^2 - \frac{M^2}{N^2} - \frac{2HM}{N}\right] = -\frac{N\beta}{2}\left(\lambda - \frac{M}{N}\right)^2 + \frac{\beta M^2}{2N} + \beta H M \\
&= -\frac{N\beta}{2}\left(\lambda - \frac{M}{N}\right)^2 - \beta\mathcal{H} \ ,
\end{aligned}$$

where we have used that

$$\mathcal{H} = -\frac{M^2}{2N} - HM \ .$$

This allows us to compute the Gaussian integral in (2) as

$$\begin{aligned}
&\sqrt{\frac{N\beta}{2\pi}}\int d\lambda \exp\left(-\frac{N\beta\lambda^2}{2} + \sum_i \beta\,(\lambda+H)\,s_i\right) \\
&\qquad = \sqrt{\frac{N\beta}{2\pi}}\int d\lambda \exp\left(-\frac{N\beta}{2}\left(\lambda - \frac{M}{N}\right)^2\right)\exp\left(-\beta\mathcal{H}\right) = \sqrt{\frac{N\beta}{2\pi}}\sqrt{\frac{2\pi}{N\beta}}\,\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{-\beta\mathcal{H}} \ .
\end{aligned}$$

(b) Show that the partition function can be written as

$$Z = \sqrt{\frac{N\beta}{2\pi}}\int d\lambda\, \mathrm{e}^{-N\beta A(\lambda)} \ , \qquad A(\lambda) = \frac{\lambda^2}{2} - \frac{1}{\beta}\ln\left(2\cosh\left[\beta(\lambda + H)\right]\right) \tag{3}$$

Solution. Let's calculate the partition function:

$$Z = \sum_{\text{configurations}} \mathrm{e}^{-\beta\mathcal{H}} = \sum \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \exp\left\{ -\frac{N\beta\lambda^2}{2} + \sum_i \beta\left(\lambda + H\right) s_i \right\}$$

$$= \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \mathrm{e}^{-\frac{N\beta\lambda^2}{2}} \sum \mathrm{e}^{\beta \sum (\lambda+H) s_i} \ . \quad \text{(S.1)}$$

We recognize the last sum as the partition function of an Ising paramagnet with noninteracting spins in a magnetic field $\lambda + H$. As a reminder:

$$\sum_{\{s_i\}} \mathrm{e}^{\beta \sum_i (\lambda+H) s_i} = \sum_{\{s_i\}} \prod_i \mathrm{e}^{\beta(\lambda+H)s_i} = \left(\sum_{s=\pm 1} \mathrm{e}^{\beta(\lambda+H)s} \right)^N = \left(2\cosh\left[\beta\left(\lambda+H\right)\right]\right)^N \ ,$$

so that we eventually get from (S.1),

$$\text{(S.1)} = \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \mathrm{e}^{-\frac{N\beta\lambda^2}{2}} \left(2\cosh\left[\beta\left(\lambda+H\right)\right]\right)^N = \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \mathrm{e}^{-N\beta A(\lambda)} \ ,$$

where we now have defined

$$A\left(\lambda\right) = \frac{\lambda^2}{2} - \frac{1}{\beta} \ln\left(2\cosh\left[\beta\left(\lambda+H\right)\right]\right) \ .$$

In order to determine the partition function, we will use the steepest descent method (a.k.a. Laplace method or saddle point approximation): the integral of the exponential is dominated by the maximum of the function in the exponential. Technically this is done by expanding the function in the exponent to second order at its maximum, and neglecting further orders.

(c) Determine the equation that λ should satisfy in order for it to be the maximum of the argument of the exponential.

Show that the partition function can be written (for large N) as

$$Z \approx \mathrm{e}^{-N\beta f} \ ; \qquad f = A\left(\lambda_0\right) + \frac{1}{2N\beta} \ln A''\left(\lambda_0\right) \approx A\left(\lambda_0\right) \ , \qquad (4)$$

where f is the free energy per spin and λ_0 is the minimizer of the function $A\left(\lambda\right)$.

Solution. In order to apply Laplace's method to the calculation of the partition function (3), we first need to determine the maximum of the argument of the exponential in the integral. This corresponds to finding the minimum of the function $A\left(\lambda\right)$. The condition of the minimum is

$$\frac{\partial A}{\partial \lambda} = 0 \ . \qquad \text{(S.2)}$$

Differentiating the expression of $A\left(\lambda\right)$ given by (3),

$$A'\left(\lambda\right) := \frac{\partial A}{\partial \lambda} = \lambda - \left\{ \frac{1}{\beta} \frac{1}{2\cosh\left[\beta\left(\lambda+H\right)\right]} \times 2\sinh\left[\beta\left(\lambda+H\right)\right] \times \beta \right\} = \lambda - \tanh\left[\beta\left(\lambda+H\right)\right] \ .$$

Thus the minimum condition (S.2) for $A\left(\lambda\right)$ is simply

$$\lambda = \tanh\left[\beta\left(\lambda+H\right)\right] \ . \qquad \text{(S.3)}$$

We can now apply Laplace's method to approximate the partition function, by expanding the argument of the exponential to second order and neglecting further orders. Of course the first order is zero because the expansion is done at a stationary point. Let λ_0 be the minimum, satisfying Eq. (S.3).

$$Z \approx \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \mathrm{e}^{-N\beta A(\lambda_0) - \frac{N\beta}{2} A''(\lambda_0)(\lambda-\lambda_0)^2} = \sqrt{\frac{N\beta}{2\pi}} \, \mathrm{e}^{-N\beta A(\lambda_0)} \sqrt{\frac{2\pi}{N\beta A''\left(\lambda_0\right)}} = \frac{\mathrm{e}^{-N\beta A(\lambda_0)}}{\sqrt{A''\left(\lambda_0\right)}} = \mathrm{e}^{-N\beta f} \ ,$$

where the free energy per spin $f = F/N$ is given by

$$f = A(\lambda_0) - \frac{1}{2N\beta} \ln A''(\lambda_0) \approx A(\lambda_0) \ .$$

(d) Show that λ_0 is precisely the average magnetization of a spin, $\lambda_0 = \langle s_i \rangle =: m$. Deduce that your result coincides with the magnetization that you would get via mean field theory.

Solution. The magnetization is given by

$$m = -\frac{\partial f}{\partial H} = -\frac{\partial A}{\partial H} - \frac{\partial A}{\partial \lambda}\frac{\partial \lambda_0}{\partial H} = -\frac{\partial A}{\partial H} \ , \tag{S.4}$$

recalling that λ_0 also depends on H, but that $\frac{\partial A}{\partial \lambda}$ vanishes at its minimum, killing the second term in the expression above. Differentiating $A(\lambda)$ in (3) by H,

$$\frac{\partial A}{\partial H} = -\frac{1}{\beta}\frac{1}{2\cosh[\beta(\lambda+H)]} \times 2\sinh[\beta(\lambda+H)] \times \beta = -\tanh[\beta(\lambda+H)] \ ,$$

which, at $\lambda = \lambda_0$, is in virtue of Eq. (S.3) simply

$$\left.\frac{\partial A}{\partial H}\right|_{\lambda_0} = -\tanh[\beta(\lambda_0+H)] = -\lambda_0 \ .$$

This means that the magnetization is

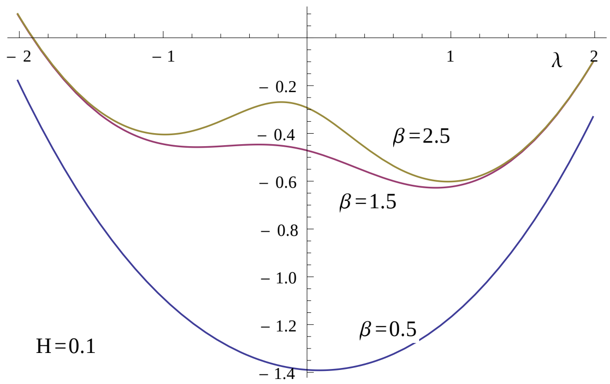
$$m = (\text{S.4}) = -\left.\frac{\partial A}{\partial H}\right|_{\lambda_0} = \lambda_0 \ . \tag{S.5}$$

Thus, the magnetization m obeys

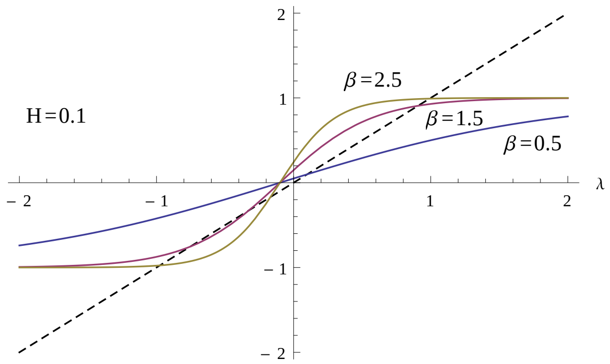
$$m = \tanh[\beta(m+H)] \ . \tag{S.6}$$

(e) When applying the Laplace method to calculate the partition function λ_0 must be the global minimizer of the function $A(\lambda)$. Convince yourself that for $H \neq 0$ the function $A(\lambda)$ has a unique global minimum. Do so by plotting $A(\lambda)$ for a fixed $H \neq 0$ and for different β. What is the error in the approximation if there are two identical minima?

Solution. In general, the function A will have one or two local minima, and if $H \neq 0$, one will be lower than the other (see Figure below). If the two are the same, the error done in the approximation of Z is a factor 2, which affects the free energy as an additive constant only. A plot of $A(\lambda)$:



The zeros of $A(\lambda)$ can rather easily be found geometrically as the intersection of the lines $y = \lambda$ and $y = \tanh[\beta(\lambda+H)]$:


H=0.1
β=2.5
β=1.5
β=0.5
λ
2
1
−1
−2
−2
−1
1
2