Exercise 1. ***Tight-binding model***

Consider non-interacting particles on a lattice of N sites with periodic boundary condition, i.e. on a discrete ring. The position variable becomes a discrete variable $\vec{r} \to x_i$ and the field operators with spin label s, $\Psi_s^{(\dagger)}(\vec{r})$, become $\Psi_{s,i}^{(\dagger)} \equiv \Psi_s^{(\dagger)}(x_i)$.

(a) Find the eigensolutions of the problem for the Hamiltonian

$$\mathcal{H} = -t \sum_s \sum_{j=0}^{N-1} \left(\Psi_{s,j+1}^\dagger \Psi_{s,j} + \Psi_{s,j}^\dagger \Psi_{s,j+1} \right) , \tag{1}$$

by the use of the Fourier transform of the field operators

$$a_{s,k} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-ijk} \Psi_{s,j} , \qquad k \in \left\{ \frac{2\pi}{N} \left(n - \left\lfloor \frac{N-1}{2} \right\rfloor \right) \middle| n = 0, 1, 2, \ldots, N-1 \right\} , \tag{2}$$

where $\lfloor x \rfloor$ denotes the integer part of x. Write the result in occupation number basis of the eigenstates. How can the terms $\Psi_{s,j+1}^\dagger \Psi_{s,j}$ be interpreted?

(b) Given the particles are fermionic, the transformation

$$b_{s,k} = \frac{1}{\sqrt{N}} \sum_j e^{-ijk} \Psi_{s,j}^\dagger \tag{3}$$

diagonalizes the Hamiltionian as well. Rewrite the problem in the occupation number basis of the $b_k^{(\dagger)}$ operators. What is the difference between the two formulations, how are they related?

(c) Consider now a fixed number of M particles to be in the system. Calculate the leading order of the entropy in the high temperature expansion $T \to \infty$. Compare it to the case of free fermions. Can you recover the particle-hole symmetry in the result?

(d) Find the magnetic susceptibility using the fluctuation-dissipation theorem

$$\chi = \frac{1}{N} \frac{1}{k_B T} \left[\langle M_z^2 \rangle - \langle M_z \rangle^2 \right] , \tag{4}$$

where the magnetization operator is defined by

$$M_z = \frac{g\mu_B}{\hbar} \sum_j S_j = \mu_B \sum_{j=0}^{N} \sum_{s=\pm 1} s \Psi_{s,j}^\dagger \Psi_{s,j} . \tag{5}$$

Determine the result in the low-temperature limit by taking $N \to \infty$.

Hint: Rewrite the magnetization operator in occupation basis and use the Fermi-Dirac distribution.

(e*) Restricting the problem to spinless Fermions and turning on a magnetic field (introduced in a specific gauge) perpendicular to the ring, changes the Hamiltonian to

$$\mathcal{H} = -t \sum_{j=0}^{N-1} \left(e^{-i\varphi} \Psi_{j+1}^\dagger \Psi_j + e^{i\varphi} \Psi_j^\dagger \Psi_{j+1} \right) . \tag{6}$$

In this case, calculate the expectation value of the current density operator

$$j = \frac{1}{N} \sum_n j_n , \qquad j_n = -i \left(\Psi_{n+1}^\dagger \Psi_n - \Psi_n^\dagger \Psi_{n+1} \right) . \tag{7}$$

Interpret the current density operator in terms of particles hopping from site to site.

Exercise 2. ***Exact solution of the Ising chain***

In this exercise we will investigate the physics of one of the few *exactly solvable interacting* models, the one-dimensional Ising model (Ising chain). Consider a chain of $N+1$ Ising-spins with free ends and nearest neighbor coupling $-J$ ($J > 0$ for ferromagnetic coupling)

$$\mathcal{H}_{N+1} = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}, \quad \sigma_i = \pm 1. \tag{8}$$

We are interested in the thermodynamic limit of this system, i.e. we assume N to be very large.

(a) Compute the partition function Z_{N+1} using a recursive procedure.

(b) Find expressions for the free energy and entropy, as well as for the internal energy and heat capacity. Compare your results to the ideal paramagnet.

(c) Calculate the magnetization density $m = \langle \sigma_j \rangle$ where the spin σ_j is not close to either end of the chain. Which symmetries does the system exhibit? Interpret you result in terms of symmetry arguments.

(d*) Show that the *spin correlation function* $\Gamma_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ decays exponentially with increasing distance $|j - i|$ on the scale of the so-called *correlation length* ξ, i.e. $\Gamma_{ij} \sim \mathrm{e}^{-|j-i|/\xi}$. Show that $\xi = -[\log(\tanh \beta J)]^{-1}$ and interpret your result in the limit $T \to 0$.

(e*) Calculate the magnetic susceptibility in zero magnetic field using the fluctuation-dissipation relation of the form

$$\frac{\chi(T)}{N} = \frac{1}{k_\mathrm{B} T} \sum_{j=-N/2}^{N/2} \Gamma_{0j}, \tag{9}$$

in the thermodynamic limit, $N \to \infty$. For simplicity we assume N to be even. Note that $\chi(T)$ is defined to be extensive, such that we obtain the intensive quantity by normalization with N.

(f*) Approximate $1/\chi(T)$ up to first order in $2\beta J$ in the high-temperature limit ($\beta \to 0$). Use this result to calculate the *Weiss temperature* Θ_W, which is defined by $1/\chi(\Theta_\mathrm{W}) = 0$.

Office Hours: Monday, November 24, 8–10 AM (Roman Süsstrunk, HIT K 23.7).