Exercise 1. ***Ideal fermionic quantum gas in a harmonic trap***

In this exercise we study the fermionic spinless ideal gas confined in a three-dimensional harmonic potential and compare it with the classical case (see Exercise Sheet 3). The eigenenergies of the gas are given by

$$\varepsilon_{\mathbf{a}} = \hbar\omega(a_x + a_y + a_z)\,, \tag{1}$$

where $\mathbf{a} = (a_x, a_y, a_z)$, with $a_i \in \{0, 1, 2, ...\}$, labels the states and the zero point energy $\varepsilon_0 = 3\,\hbar\omega/2$ was omitted. The occupation number corresponding to state $\mathbf{a}$ is given by $n_{\mathbf{a}}$.

(a) Consider the high-temperature, low-density limit ($z \ll 1$). Derive the grand canonical partition function $\mathcal{Z}_f$ of this system and compute the grand potential Ω_f. Show that

$$\Omega_f \propto f_4(z)\;, \tag{2}$$

where the function $f_s(z)$ is defined as

$$f_s(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^s}\;. \tag{3}$$

(b) Calculate the particle number $\langle N \rangle$ and the internal energy U as a function of N. In order to get U in terms of N (instead of dealing with the chemical potential), introduce the parameter

$$\rho \equiv \left(\frac{\hbar\omega \langle N \rangle^{1/3}}{k_B T} \right)^3 \tag{4}$$

and relate it to z using the high-temperature, low-density expansion of $\langle N \rangle$ (up to $\mathcal{O}(z^2)$). Interpret the condition $\rho \ll 1$.

Finally, expand U up to second order in ρ, relating it to N.

(c) Compute the heat capacity C. Which quantity has to be fixed in order to do this?

(d) Compute the isothermal compressibility κ_T.

(e) Interpret your results for U, C, and κ_T by comparing them with the corresponding results for the classical Boltzmann gas. How do the quantum corrections influence the fermionic system?

Exercise 2. ***Sommerfeld expansion and density of states***

Consider a thermally equilibrated system of non-interacting fermions with single particle states labeled by the quantum numbers ν and corresponding energies ε_ν.

(a) Work in the grand canonical ensemble and write the particle and energy densities in the form

$$n = \frac{1}{V}\sum_\nu f(\varepsilon_\nu) = \int d\varepsilon\, g(\varepsilon) f(\varepsilon)\,, \tag{5}$$

$$u = \frac{1}{V}\sum_\nu \varepsilon_\nu f(\varepsilon_\nu) = \int d\varepsilon\, \varepsilon g(\varepsilon) f(\varepsilon)\,, \tag{6}$$

where $f(\varepsilon)$ is the Fermi-Dirac distribution function. What is $g(\varepsilon)$?

(b) The above expressions for n and u are of the form

$$\int_{-\infty}^{\infty} d\varepsilon\, H(\varepsilon) f(\varepsilon)\,. \tag{7}$$

For temperatures $T \ll \frac{\varepsilon_F}{k_B}$ (which is typically the case for metals), $H(\varepsilon)$ is slowly varying in the region where $\frac{df}{d\varepsilon} \neq 0$ significantly and the Sommerfeld expansion[1]

$$\int_{-\infty}^{\infty} d\varepsilon\, H(\varepsilon) f(\varepsilon) = \int_{-\infty}^{\mu} d\varepsilon\, H(\varepsilon) + \frac{\pi^2}{6}(k_B T)^2 H'(\mu) + \frac{7\pi^4}{360}(k_B T)^4 H'''(\mu) + \mathcal{O}\left(\frac{k_B T}{\mu}\right)^6 \tag{8}$$

becomes handy. Make use of this expansion up to $\mathcal{O}\left(\frac{k_B T}{\mu}\right)^2$ to expand n and u in T.

Hint: Use (in a self-consistent way) that $\mu - \varepsilon_F \propto T^2$ in leading order in T and expand

$$\int_{-\infty}^{\mu} d\varepsilon\, H(\varepsilon) \approx \int_{-\infty}^{\varepsilon_F} d\varepsilon\, H(\varepsilon) + (\mu - \varepsilon_F) H(\varepsilon_F)\,. \tag{9}$$

(c) Find the chemical potential μ and the specific heat c_v at constant density n.

(d) Determine $g(\varepsilon)$ for the case of a free Fermi gas and calculate its chemical potential and specific heat from the previous results. Compare your result for the specific heat with the one for a classical gas.

(e) For the free Fermi gas $g'(\varepsilon_F) > 0$. This does not need to be the true in more complex systems such as solids (cf., e.g., semiconductors). What are the consequences of $g'(\varepsilon_F) \leq 0$?

Office Hours: Monday, October 27, 8–10 AM (Roman Süsstrunk, HIT K 23.7).

[1] For a reference on the Sommerfeld expansion see, e.g., Ashcroft, N. W. and Mermin N. D., *Solid State Physics*, Holt, Rinehart and Winston, 1976.