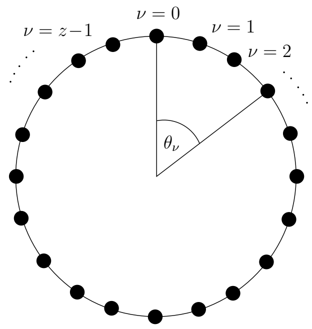
Statistical Physics
Exercise Sheet 1

Exercise 1.1 Statistical Polarization

We study a system of N compasses with a needle pointing in one of z directions. The needle hops to a neighboring direction with a rate Γ (nearest-neighbor interaction). We assume that z is even for ease of presentation.



a) Express the master equation and the entropy (H-function) of this system in terms of the numbers N_ν of compasses pointing in direction ν, where $\nu \in \{0, 1, \ldots, z-1\}$. What is the equilibrium state of this system?

b) The system is prepared with all the needles pointing uniformly distributed in the directions $\nu \in \{0, 1, \ldots, z/2-1\}$. What is the entropy of the initial distribution? Compare your result with the equilibrium entropy and interpret the result with respect to the ideal gas entropy.

c) At time $t = 0$, all needles point in the same direction ($N_0 = N$, $N_\nu = 0\ \forall \nu \neq 0$). Calculate for long times ($t \gg z^2/\Gamma$) the polarization of the system

$$P(t) := \langle \cos\theta \rangle(t) = \sum_\nu \frac{N_\nu(t)}{N} \cos(\theta_\nu), \quad \theta_\nu = \frac{2\pi\nu}{z}, \tag{1}$$

and compare the relaxation of the polarization with the one for the entropy.

Hint: Use the discrete Fourier transform

$$N_\nu(t) = \sum_{k=-z/2}^{z/2-1} \tilde{N}_k(t)\, e^{-i\frac{2\pi}{z}k\nu},$$

and express the master equation and $P(t)$ in terms of the coefficients $\tilde{N}_k(t)$.

d) Starting with the same initial distribution as in c), calculate the exact time dependence of N_ν for the case of $z \to \infty$.

Hint: Express N_ν using a reformulation of the Jacobi–Anger expansion in terms of the modified Bessel functions of the first kind $I_n(x)$,

$$e^{x\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(x) e^{in\theta}. \tag{2}$$

e) We go to a continuum description: $z \to \infty$ with $L = za$ constant, where a is the distance between points. We find the diffusion equation

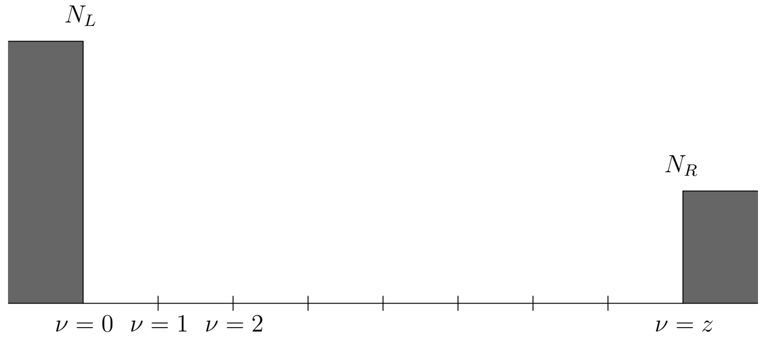
$$\dot{\rho}(x,t) = D\,\partial_x^2 \rho(x,t). \tag{3}$$

Determine D in this equation.

Starting with the same initial distribution as in c) (that is, the corresponding continuum limit), calculate the time dependence of the entropy for $L \to \infty$ (i.e., ignoring boundary effects). Interpret your result with respect to the ideal gas.

Exercise 1.2 Particle Current and Entropy Production

We consider a chain with z sites and nearest-neighbor interactions with rate Γ (analogous to Exercise Exercise 1.1) that is connected to two reservoirs such that the boundary conditions $N_0 = N_L$ and $N_z = N_R$ hold for all times (see figure).



a) Find the stationary state of the chain, and compute the corresponding particle current J_ν on the bond between sites $\nu - 1$ and ν, which is defined through the continuity equation $\dot{N}_\nu = J_\nu - J_{\nu+1}$.

b) Show that while the H-function of the chain, $H(t) = -\sum_{\nu=0}^{z} \frac{N_\nu}{N} \log \frac{N_\nu}{N}$, is constant in the steady state, the entropy production as given by

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{k_B N}{2} \sum_{\nu,\nu'} \Gamma_{\nu\nu'} \Big(\ln \frac{N_\nu}{N} - \ln \frac{N_{\nu'}}{N} \Big) \Big(\frac{N_\nu}{N} - \frac{N_{\nu'}}{N} \Big) \tag{4}$$

is positive. Discuss this result. What process is responsible for the increase in entropy?

c) Consider the stationary state close to the equilibrium, where

$$N_L - N_R = \epsilon, \qquad 0 < \epsilon \ll N_L.$$

Show that while the currents are proportional to ϵ, the entropy production is proportional to ϵ^2. What does that mean for the description of transport phenomena (heat, particles) close to equilibrium?

Office Hours: Monday, September 29, 8–10 AM (Roman Süsstrunk, HIT K 23.7).