Quantum computation with Josephson junctions
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Follows closely
Outline

• Josephson Hamiltonian
• Quantization of Josephson Hamiltonian
• Phase qubit controlled by current
• Phase qubit controlled by flux
• Capacitive coupling of qubits
• Inductive coupling of qubits
Isolated Josephson junction

- Two classical variables:
  - Particle number \(2p = n_b - n_a\)
  - Phase \(\delta = \delta_b - \delta_a\)
- Capacity: \(V = \frac{Q}{C} = \frac{2ep}{C}\)

Josephson junctions can be described with two classical variables:

- Particle number
- Phase

These variables are characterized by Cooper pairs, which tunnel through the junction. When isolated, we can define the number of Cooper pairs \(n_a, n_b\) and the quantum phases \(\delta_a, \delta_b\) of the Cooper pair wave functions on the two sides of the junction.

The charge and phase differences \(n_a - n_b\) and \(\delta = \delta_b - \delta_a\) are the essential parameters describing the properties of the Josephson junction.

Josephson has shown in 1962 that a current \(I_J = I_0 \sin \delta\) circulates across the junction without applied voltage if \(\delta\) is constant and that an ac current of frequency \(\nu = \frac{2eV}{\hbar}\) oscillates between the two sides if a voltage \(V\) is imposed on the junction.

These fundamental results can be derived either from the heuristic Ginsburg-Landau model of superconductivity, or from the more fundamental Bardeen Cooper Shrieffer (BCS) theory. They can also be understood by a simple model describing the ensemble of Cooper pairs as a gas of composite bosons tunneling between two potential wells separated by a barrier (similar Josephson effects have been recently demonstrated in Bose-Einstein condensates made of cold atoms).

- Josephson's equations:
  - DC Josephson effect:
    \[I_J = -2e \frac{dp}{dt} = I_0 \sin \delta\]
  - AC Josephson effect:
    \[\frac{d\delta}{dt} = \frac{2eV}{\hbar} = \frac{4e^2p}{\hbar C}\]
- Looks like a pendulum!
Josephson Hamiltonian

\[ H = E_C p^2 - E_J \cos \delta; \quad E_C = \frac{2e^2}{C}, \quad E_J = \frac{\hbar I_0}{2e} \]

- Convince yourself by obtaining the equations of motion:
  \[
  \frac{dp}{dt} = -\frac{1}{\hbar} \frac{\partial H}{\partial \delta} = -\frac{I_0}{2e} \sin \delta \\
  \frac{d\delta}{dt} = \frac{1}{\hbar} \frac{\partial H}{\partial p} = \frac{4e^2 p}{\hbar C}
  \]

- \( p \) and \( \delta \) are canonically conjugate variable!

Linearized version is a “lump circuit”

\[ H_l = E_C p^2 + E_J \frac{\delta^2}{2} \]

\( E_C \) ... capacitive energy

\( E_J \) ... inductive energy
Quantizing the Josephson Hamiltonian

- Since \( p \) and \( \delta \) are canonically conjugate, we can postulate the commutator:
  \[
  [p, \delta] = i
  \]
- Quantized energy levels (not harmonic!)
  \[
  \omega_{01} \approx \sqrt{2E_J E_C}
  \]
- Particle Number \( (p) \) and phase \( (\delta) \) cannot be measured exactly at the same time!
- Uncertainty relationship:
  \[
  \langle \Delta p^2 \rangle = \sqrt{\frac{E_J}{8E_C}} , \quad \langle \Delta \delta^2 \rangle = \sqrt{\frac{E_C}{2E_J}}
  \]
- Two limiting cases:
  \[
  \frac{E_J}{E_C} \gg 1 \quad \text{...phase well defined} \quad \Rightarrow \text{phase qubit}
  \]
  \[
  \frac{E_J}{E_C} \ll 1 \quad \text{...charge well defined} \quad \Rightarrow \text{charge qubit}
  \]
Realising a superconducting Qubit

Prerequisites:
• controlled manipulation of qubit without disturbing adjacent elements
• controlled inter-qubit coupling
• detection of qubit state
• limited influence of external environment
• sufficiently long dephasing and decoherence times

Josephson Qubits:
• Qubit Hamiltonian adjustable by bias current and flux
• State preparation via rf-Pulses
• inter-qubit coupling achieved by capacitive or inductive coupling
Phase Qubit controlled by current

Josephson junction driven by constant current:
\[ I = I_0 \sin \delta + \frac{dQ}{dt} = I_0 \sin \delta + 2e \frac{dp}{dt} \]

modified Hamiltonian:
\[ H(I) = \frac{2e^2}{C} p^2 - \frac{\hbar}{2e} (I \delta + I_0 \cos \delta) \]

washboard potential:

Frequency \( \omega_{01} \) can be tuned by current \( I \):

State detection:
- Increase \( I \) until the tunnel barrier is too low for state 1.
  - If state 1 \( \rightarrow \) critical value of \( \frac{d\delta}{dt} \) exceeded \( \rightarrow \) normal metal state \( \rightarrow \) voltage drop
Phase qubit controlled by flux

**Effect of the magnetic flux:**
- Phase jump at junction:

\[
\frac{2e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi \frac{\Phi}{\Phi_0} = \delta \quad ; \quad \Phi_0 = \frac{\hbar}{2e}
\]

- Shielding current I

\[\Phi = \Phi_e - LI\]

Add magnetic energy \(\frac{1}{2} LI^2\) to the Hamiltonian:

\[
H = \frac{2e^2}{C} p^2 + \frac{\Phi_0^2}{2L} \left( \frac{\delta}{2\pi} - \frac{\Phi_e}{\Phi_0} \right)^2 - \frac{\Phi_0}{2\pi} I_0 \cos \delta
\]

**Symmetric case** \(\Phi_e = (n + \frac{1}{2})\Phi_0\), flux qubit:

\[
\Phi_e = \frac{\Phi_0}{2}
\]

**Otherwise asymmetric, phase qubit:**

\[
\Phi_e = \frac{\Phi_0}{4}
\]
Detecting qubit state

Manipulation of flux $\Phi_e$ until qubit transits from one well to the other:

\[ U(\delta) \]

$\delta$ changes of order $2\pi$ -> $\Phi_e$ changes of order $\Phi_0$

Flux change detected by inductively coupled SQUID:

Manipulation of qubit state by an AC current with frequency close to $\omega_{01}$:

\[ H(t) = H_0 + \frac{\phi_0}{2\pi} \delta I_{rf} \sin(\omega t - \phi) \]

-> Rabi oscillations -> Rotation of qubit state!
Capacitive coupling of two phase qubits

Two identical phase qubits A and B coupled by $C_X \ll C$:

- Achieve tuneable coupling through frequency selection with bias current

Coupling energy:

$$H_{int} = \frac{1}{2} C_X (V_A - V_B)^2 = \frac{2e^2 C_X}{C^2} (p_A - p_B)^2$$

Regrouping terms:

$$H = H_A + H_B + H_{int} = H'_A + H'_B + H'_{int}$$

with

$$H'_i = \frac{2e(C + C_X)}{C^2} p_i^2 + U(\delta_i)$$

and

$$H'_{int} = -\frac{4eC_X^2}{C^2} p_A p_B$$
Coupling of two flux qubits

For flux qubits the coupling can be achieved by the magnetic dipole-dipole interaction

- Can be coupled over long distances by “flux transformers”
- Coupling through SQUID can be tuned

Conclusions and Outlook

• charge and phase of Josephson junction are conjugate variables

• Frequency and detection of qubit achieved by current/flux bias

• Manipulation of qubit state by rf-pulses

• Coupled qubits can be used to create entanglement and realise quantum gates

• Decoherence can be modelled by complex impedance

• Coupling of qubit to rf LC Resonator -> Jaynes-Cummings Hamiltonian -> "Circuit QED"
Thank you for your attention.