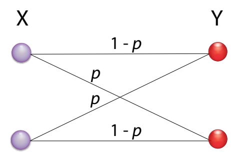
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Quantum Information Theory
Solutions 5

FS 14
Dr. J. M. Renes

Exercise 5.1 Classical channels as TPCPMs.

a) *Take the binary symmetric channel* $\mathbf{p}$*,*



Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1 - q$*, we may express this as the 1-qubit mixed state* $\rho_X = q\ |0\rangle\langle 0| + (1-q)\ |1\rangle\langle 1|$*.*

What is the quantum state ρ_Y *that represents the final probability distribution* P_Y *in the computational basis?*

We have

$$
\begin{aligned}
P_Y(0) &= \sum_x P_X(x) P_{Y|X=x}(0) = q(1-p) + (1-q)p \\
P_Y(1) &= qp + (1-q)(1-p),
\end{aligned}
$$

which can be expressed as a quantum state $\rho_y = [q(1-p) + (1-q)p]\ |0\rangle\langle 0| + [qp + (1-q)(1-p)]\ |1\rangle\langle 1| \in \mathcal{L}(\mathcal{H}_Y)$.

b) *Now we want to represent the channel as a map*

$$
\begin{aligned}
\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) &\mapsto \mathcal{S}(\mathcal{H}_Y) \\
\rho_X &\to \rho_Y.
\end{aligned}
$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ *is a decomposition* $\{E_k\}_k$ *of operators* $E_k \in Hom(\mathcal{H}_X, \mathcal{H}_Y)$*,* $\sum_k E_k {E_k}^\dagger = \mathbb{1}$*, such that*

$$
\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.
$$

Find an operator-sum representation of $\mathcal{E}_{\mathbf{p}}$*.*

We take four operators, corresponding to the four different "branches" of the channel,

$$
\begin{aligned}
E_{0\to 0} &= \sqrt{1-p}|0\rangle\langle 0| \\
E_{0\to 1} &= \sqrt{p}|1\rangle\langle 0| \\
E_{1\to 0} &= \sqrt{p}|0\rangle\langle 1| \\
E_{1\to 1} &= \sqrt{1-p}|1\rangle\langle 1|.
\end{aligned}
$$

To check that this works for the classical state ρ_X, we do

$$\begin{aligned}
\mathcal{E}(\rho_X) =& \sum_{xy} E_{x\to y}\ \rho_X\ E_{x\to y}^\dagger \\
=& \sum_{xy} E_{x\to y}\ \Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]\ E_{x\to y}^\dagger \\
=&(1-p)\ |0\rangle\langle 0|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|0\rangle\langle 0| \\
&+ p\ |1\rangle\langle 0|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|0\rangle\langle 1| \\
&+ p\ |0\rangle\langle 1|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|1\rangle\langle 0| \\
&+ (1-p)\ |1\rangle\langle 1|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|1\rangle\langle 1| \\
=&q(1-p)\ |0\rangle\langle 0| \\
&+ qp\ |1\rangle\langle 1| \\
&+ (1-q)p\ |0\rangle\langle 0| \\
&+ (1-q)(1-p)\ |1\rangle\langle 1| = \rho_Y .
\end{aligned}$$

c) *Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state ρ_X is not diagonal in the computational basis?*

In general, we can express the state in the computational basis as $\rho_X = \sum_{ij} \alpha_{ij}|i\rangle\langle j|$, with the usual conditions (positivity, normalization). Applying the map gives us

$$\begin{aligned}
\mathcal{E}(\rho_X) =& \sum_{xy} E_{x\to y}\ \Big[\sum_{ij}\alpha_{ij}|i\rangle\langle j|\Big]\ E_{x\to y}^\dagger \\
=&(1-p)\ |0\rangle\langle 0|\Big[\sum_{ij}\alpha_{ij}|i\rangle\langle j|\Big]|0\rangle\langle 0| \\
&+ p\ |1\rangle\langle 0|\Big[\sum_{ij}\alpha_{ij}|i\rangle\langle j|\Big]|0\rangle\langle 1| \\
&+ p\ |0\rangle\langle 1|\Big[\sum_{ij}\alpha_{ij}|i\rangle\langle j|\Big]|1\rangle\langle 0| \\
&+ (1-p)\ |1\rangle\langle 1|\Big[\sum_{ij}\alpha_{ij}|i\rangle\langle j|\Big]|1\rangle\langle 1| \\
=&\alpha_{11}(1-p)\ |0\rangle\langle 0| + \alpha_{11}p\ |1\rangle\langle 1| \\
&+ \alpha_{22}p\ |0\rangle\langle 0| + \alpha_{22}(1-p)\ |1\rangle\langle 1|.
\end{aligned}$$

Using $\alpha_{11} := \alpha, \alpha_{22} = 1-\alpha$, we get $\mathcal{E}(\rho_X) = [\alpha(1-p) + (1-\alpha)p]\ |0\rangle\langle 0| + [\alpha p + (1-\alpha)(1-p)]\ |1\rangle\langle 1|$. The channel ignores the off-diagonal terms of ρ_X: it acts as a measurement on the computational basis followed by the classical binary symmetric channel.

d) *Now consider an arbitrary classical channel* $\mathbf{p}$ *from an n-bit space X to an m-bit space Y, defined by the conditional probabilities* $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ *as a map* $\mathcal{E}_\mathbf{p} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ *in the operator-sum representation.*

We generalize the previous result as

$$\begin{aligned}\mathcal{E}_{\mathbf{p}}(\rho_X) &= \sum_{x,y} P_{Y|X=x}(y)\ |y\rangle\langle x|\rho_X|x\rangle\langle y| \\ &= \sum_{x,y} E_{x\to y}\rho_X E^\dagger x\to y, \quad E_{x\to y} = \sqrt{P_{Y|X=x}(y)}\ |y\rangle\langle x|.\end{aligned}$$

To see that this works, take a classical state $\rho_X = \sum_x P_X(x)\ |x\rangle\langle x|$ as input,

$$\begin{aligned}\mathcal{E}_{\mathbf{p}}(\rho_X) &= \sum_{x,y} P_{Y|X=x}(y)\ |y\rangle\langle x|\Big(\sum_{x'} P_X(x')\ |x'\rangle\langle x'|\Big)|x\rangle\langle y| \\ &= \sum_{x,y} P_{Y|X=x}(y)\ P_X(x)\ |y\rangle\langle y| \\ &= \sum_{y} P_y(y)\ |y\rangle\langle y|.\end{aligned}$$

Exercise 5.2 Different Quantum Channels

Consider two single-qubit Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a CPTP map

$$\begin{aligned}\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B) \\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho,\end{aligned}$$

which is called depolarizing channel*.*

a) *Find a Kraus representation for $\mathcal{E}_p$.*

For simplicity of notation, we denote the Pauli matrices by X, Y, Z.

Remembering that $X^2 = Y^2 = Z^2 = \mathbb{1}$, $XY = -YX = Z$, $YZ = -ZY = X$ and $ZX = -XZ = Y$, you can verify that

$$\mathbb{1} = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z).$$

From this follows the operator sum representation $\{M_x\}_x$,

$$M_1 = \sqrt{1-\frac{3p}{4}}\ \mathbb{1}, \quad M_2 = \frac{\sqrt{p}}{2}X, \quad M_3 = \frac{\sqrt{p}}{2}Y, \quad M_4 = \frac{\sqrt{p}}{2}Z.$$

b) *What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? What is the physical interpretation of this?*

Using the result from part *a)* we have

$$\begin{aligned}\mathcal{E}(\rho) &= \frac{p}{2}\mathbb{1} + (1-p)\ \rho \\ &= \frac{1}{2}(\mathbb{1} + (1-p)\ \vec{r}\cdot\vec{X})\end{aligned}$$

Thus, points on a sphere with radius r are mapped to a smaller sphere with radius $(1-p)r$ — they get more mixed in that sense. In particular, pure states will not remain pure during this CPM.

c) *Find Kraus representations for the following qubit channels*

(i) *The dephasing channel:* $\rho \to \rho' = \mathcal{E}(\rho) = (1-p)\rho + p\,\mathrm{diag}(\rho_{00}, \rho_{11})$ *(the off-diagonal elements are annihiliated with probability p).*

The dephased output is the same as measuring the state in the standard basis: $\mathrm{diag}(\rho_{00}, \rho_{11}) = \sum_{j=0}^{1} P_j \rho P_j$ for $P_j = |j\rangle\langle j|$. Thus possible Kraus operators are $A_2 = \sqrt{1-p}\mathbb{1}$, $A_j = \sqrt{p}P_j$, $j = 0, 1$. But we can find a representation with fewer Kraus operators. Notice that $\sigma_z \rho \sigma_z = \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix}$. Thus $(\rho + \sigma_z \rho \sigma_z)/2 = \mathrm{diag}(\rho_{00}, \rho_{11})$ and $\rho' = \sum_{j=0}^{1} A_j \rho A_j^\dagger$ for $A_0 = \sqrt{1-p/2}\mathbb{1}$ and $A_1 = \sqrt{p/2}\sigma_z$.

(ii) *The amplitude damping (damplitude) channel, defined by the action* $|00\rangle \to |00\rangle$, $|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$.

From the unitary action we can read off the Kraus operators since $U|\psi\rangle|0\rangle = \sum_k A_k|\psi\rangle|k\rangle$. Therefore $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$.

Exercise 5.3 The Choi Isomorphism

Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \mapsto (1-\alpha)\,\frac{\mathbb{1}_2}{2} + \alpha\,\Big(\frac{\mathbb{1}_2}{2} + \sigma_x \rho \sigma_z + \sigma_z \rho \sigma_x\Big), \qquad 0 \le \alpha \le 1, \tag{1}$$

where $\{\sigma_i\}_i$ *are Pauli matrices.*

a) *Use the Bloch representation to determine for what range of* α *these mappings are positive. What happens to the Bloch sphere?*

The two-dimensional state space $\mathcal{S}(\mathcal{H}_2)$ is isomorphic to the unit sphere on $\mathbb{R}^3$:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \qquad x^2+y^2+z^2 \le 1.$$

We apply the map to this state and get

$$\rho' = \frac{1}{2}\begin{pmatrix} 1+2\alpha x & 2\alpha z \\ 2\alpha z & 1-2\alpha x \end{pmatrix}.$$

The mapping is trace-preserving, hence it is positive if and only if the determinant of ρ' is positive for all allowed values of x, y and z. The determinant is given by

$$\begin{aligned}\det(\rho') &= \frac{1}{4}(1 - 4\alpha^2 x^2 - 4\alpha^2 z^2) \\ &\ge \frac{1}{4} - \alpha^2.\end{aligned}$$

The map is positive for $0 \le \alpha \le \frac{1}{2}$.

b) *Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:*

$$\sigma_\alpha = (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

For what range of α *is the mapping a CPM?*

We need to find an expression for $\mathcal{E} \otimes \mathcal{I}(|\Psi\rangle\langle\Psi|_{AR})$. Let us see how we can extend a TPCPM on composite systems. Suppose we have a TPCPM described by

$$\mathcal{E}(\rho_A) = \sum_i X_i \rho_A Y_i.$$

Note that this is not the operator-sum representation, as Y_i is not necessarily ${X_i}^\dagger$.

When we apply the composed map to a product of matrices of the form $M \otimes N$ (not necessarily a valid quantum state), we have

$$\begin{aligned}[\mathcal{E} \otimes \mathcal{I}](M \otimes N) &= [\mathcal{E}(M)] \otimes [\mathcal{I}(N)] \\ &= \left[\sum_i X_i M Y_i\right] \otimes [\mathbb{1} N \mathbb{1}] \\ &= \sum_i [X_i \otimes \mathbb{1}]\, M \otimes N\, [Y_i \otimes \mathbb{1}]\,.\end{aligned}$$

But any density operator can be written as $\rho_{AR} = \sum_{m,n} \alpha_{mn} M_m \otimes N_n$ and TPCMs are linear maps, so

$$\begin{aligned}[\mathcal{E} \otimes \mathcal{I}](\rho_{AR}) &= [\mathcal{E} \otimes \mathcal{I}]\left(\sum_{m,n} \alpha_{mn} M_m \otimes N_n\right) \\ &= \sum_{m,n} \alpha_{mn} [\mathcal{E} \otimes \mathcal{I}]\left(M_m \otimes N_n\right) \\ &= \sum_{m,n} \alpha_{mn} \left(\sum_i [X_i \otimes \mathbb{1}]\, M_m \otimes N_n\, [Y_i \otimes \mathbb{1}]\right) \\ &= \sum_i [X_i \otimes \mathbb{1}] \left(\sum_{m,n} \alpha_{mn} M_m \otimes N_n\right) [Y_i \otimes \mathbb{1}] \\ &= \sum_i [X_i \otimes \mathbb{1}]\, \rho_{AR}\, [Y_i \otimes \mathbb{1}]\end{aligned}$$

In the case of this particular map, we have

$$\begin{aligned}\mathcal{E}_\alpha(\rho_A) &= (1-\alpha)\ \frac{\mathbb{1}_2}{2} + \alpha\ \big(\frac{\mathbb{1}_2}{2} + \sigma_x \rho_A \sigma_z + \sigma_z \rho_A \sigma_x\big) \\ &= \frac{\mathbb{1}_2}{2} + \alpha\ \big(\sigma_x \rho_A \sigma_z + \sigma_z \rho_A \sigma_x\big) \\ &= \frac{1}{4}\big(\mathbb{1}_2 \rho_A \mathbb{1}_2 + \sigma_x \rho_A \sigma_y + \sigma_y \rho_A \sigma_y + \sigma_z \rho_A \sigma_z\big) + \alpha\ \big(\sigma_x \rho_A \sigma_z + \sigma_z \rho_A \sigma_x\big) \\ \mathcal{E}_\alpha \otimes \mathcal{I}(\rho_{AR}) &= \frac{1}{4}\big(\mathbb{1}_4 \rho_{AR} \mathbb{1}_4 + [\sigma_x \otimes \mathbb{1}_2]\rho_{AR}[\sigma_y \otimes \mathbb{1}_2] + [\sigma_y \otimes \mathbb{1}_2]\rho_{AR}[\sigma_y \otimes \mathbb{1}_2] + [\sigma_z \otimes \mathbb{1}_2]\rho_{AR}[\sigma_z \otimes \mathbb{1}_2]\big) \\ &\quad + \alpha\ \big([\sigma_x \otimes \mathbb{1}_2]\rho_{AR}[\sigma_z \otimes \mathbb{1}_2] + [\sigma_z \otimes \mathbb{1}_2]\rho_{AR}[\sigma_x \otimes \mathbb{1}_2]\big),\end{aligned}$$

with

$$\sigma_x \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_y \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \sigma_z \otimes \mathbb{1}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

in the computational basis. When we apply this map to the fully entangled state we get

$$|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{E}_\alpha \otimes \mathcal{I}(|\Psi\rangle\langle\Psi|) = \frac{1}{4}\begin{pmatrix} 1 & 2\alpha & 2\alpha & 0 \\ 2\alpha & 1 & 0 & -2\alpha \\ 2\alpha & 0 & 1 & -2\alpha \\ 0 & -2\alpha & -2\alpha & 1 \end{pmatrix}.$$

We know that because of the way the Choi isomorphism is designed, a map $\mathcal{E}$ is completely positive if and only if the matrix $\mathcal{E} \otimes \mathcal{I}\ (|\Psi\rangle\langle\Psi|)$ is non negative. In this case, the eigenvalues of that matrix are

$$\lambda_\alpha^1 = \frac{1}{4} - \alpha, \quad \lambda_\alpha^2 = \lambda_\alpha^3 = \frac{1}{4}, \quad \lambda_\alpha^4 = \frac{1}{4} + \alpha,$$

so we conclude that the map $\mathcal{E}_\alpha$ is completely positive for $0 \leq \alpha \leq \frac{1}{4}$.

c) *Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.*

Note: In this solution we use results from pages 42–45 of the script. However, we adopted a slightly different notation. Our reference system R, that is a copy of A, is called A' in the script, and our purifying system P is labeled R there. Also, in the script it is considered the extension $\mathcal{I} \otimes \mathcal{E}$, while we have $\mathcal{E} \otimes \mathcal{I}$.

The Stinespring dilation tells us that for any TPCPM $\mathcal{E}_{A\to B}$ from a system A to a system B, there is a system P where one can find an isometry U from A to $B \otimes P$ that satisfies

$$\mathcal{E}_{A\to B} : \rho_A \mapsto \mathrm{Tr}_P(U_{A\to BP}\ \rho_A\ U^*_{A\to BP}).$$

If we know how to find U, it comes in a very nice form, such that it is direct to obtain the operator-sum representation of $\mathcal{E}$ by tracing out system P. It takes four steps to find the operator-sum representation of $\mathcal{E}$:

1. We apply the Choi-Jamiolkowski isomorphism (CJI) on the map,

$$\tau : \mathcal{E}_{A\to B} \mapsto [\mathcal{E}_{A\to B} \otimes \mathcal{I}_{R\to R}]\ \ (|\Psi\rangle\langle\Psi|_{AR}) =: \sigma_{BR},$$

   obtaining a density matrix σ_{BR}, where R is a copy of A — the correspondent Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_R$, have equivalent bases, $\left\{|i\rangle_{A/R}\right\}_i$.

2. We purify σ_{BR} using an auxiliar system P,

$$|\Phi\rangle_{BRP} : \mathrm{Tr}_P(|\Phi\rangle\langle\Phi|) = \sigma_{BR}.$$

3. We apply the inverse of the CJI on $|\Phi\rangle$, and obtain a very special map,

$$\tau^{-1}(|\Phi\rangle\langle\Phi|_{BRP}) = \mathcal{U}_{A\to BP} : \rho_A \to U_{A\to BP}\ \rho_A\ U^*_{A\to BP},$$

   where U is an isometry.

4. We trace out the purifying system P and recover $\mathcal{E}$, in the operator-sum representation,

$$\mathrm{Tr}_P(\mathcal{U}_{A\to BP}) = \mathcal{E}_{A\to B}.$$

Let us now apply this protocol to our map. We had

$$\mathcal{E}_\alpha \otimes \mathcal{I}\ (|\Psi\rangle\langle\Psi|) = \frac{1}{4}\begin{pmatrix} 1 & 2\alpha & 2\alpha & 0 \\ 2\alpha & 1 & 0 & -2\alpha \\ 2\alpha & 0 & 1 & -2\alpha \\ 0 & -2\alpha & -2\alpha & 1 \end{pmatrix}.$$

This matrix has eigenvalues and eigenvectors

$$\lambda_\alpha^1 = \frac{1}{4} - \alpha, \qquad \lambda_\alpha^2 = \frac{1}{4}, \qquad \lambda_\alpha^2 = \frac{1}{4}, \qquad \lambda_\alpha^4 = \frac{1}{4} + \alpha,$$

$$|v^1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad |v^2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad |v^3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |v^4\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

The eigenvectors do not depend on α, and their Schimdt decompositions are all of the form

$$|v^i\rangle = \frac{1}{\sqrt{2}}\Big(|b_0^k\rangle \otimes |0\rangle_R + |b_1^k\rangle \otimes |1\rangle_R\Big),$$

with

$$\begin{array}{llll} |b_0^1\rangle = \frac{1}{\sqrt{2}}\Big(-|0\rangle + |1\rangle\Big), & |b_0^2\rangle = -|1\rangle, & |b_0^3\rangle = |0\rangle, & |b_0^4\rangle = \frac{1}{\sqrt{2}}\Big(-|0\rangle - |1\rangle\Big), \\ |b_1^1\rangle = \frac{1}{\sqrt{2}}\Big(\quad |0\rangle + |1\rangle\Big), & |b_1^2\rangle = |0\rangle, & |b_1^3\rangle = |1\rangle, & |b_1^4\rangle = \frac{1}{\sqrt{2}}\Big(-|0\rangle + |1\rangle\Big). \end{array}$$

This gives us

$$\begin{aligned} E_\alpha^k &= \sqrt{2\ \lambda^k}\ \sum_\ell \frac{1}{\sqrt{2}}\ |b_\ell^k\rangle_B \langle r_\ell^k|_A \\ &= \sqrt{\lambda_\alpha^k}\ \Big(|b_0^k\rangle\langle 0| + |b_1^k\rangle\langle 1|\Big) \\ &= \sqrt{\lambda_\alpha^k}\ M^k, \end{aligned}$$

for $\mathcal{E}_\alpha(\rho_A) = \sum_k \lambda^k M^k \rho_A M^{k^*}$. The matrices M^k are

$$M^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M^4 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$