Quantum Information Theory
Solutions 6

Exercise 6.1 Classical capacity of the depolarizing channel

Consider the depolarizing channel we have treated in the exercise before that is described by the CPTP map

$$\begin{aligned} E_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B) \\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho. \end{aligned}$$

a) *Now we will see what happens when we use this quantum channel to send classical information. We start with an arbitrary input probability distribution $P_X(0) = q, P_X(1) = 1-q$. We encode this distribution in a state $\rho_X = q\,|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$. Now we send ρ_X over the quantum channel, i.e., we let it evolve under $E_\mathbf{p}$. Finally, we measure the output state, $\rho_Y = E_\mathbf{p}(\rho_X)$ in the computational basis. Compute the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.*

Applying the map to this state results in

$$\begin{aligned} \mathcal{E}(\rho_X) &= \left(\frac{p}{2} + (1-p)q\right)\ |0\rangle\langle 0| + \left(\frac{p}{2} + (1-p)(1-q)\right)\ |1\rangle\langle 1| \\ &= P_Y(0)\ |0\rangle\langle 0| + P_Y(1)\ |1\rangle\langle 1|, \end{aligned}$$

so $P_Y(0) = \frac{p}{2} + (1-p)q, P_Y(1) = \frac{p}{2} + (1-p)(1-q)$. The conditional probabilities can be arranged in a transition matrix $(T)_{xy} = P_{Y|X=x}(y)$ as follows:

$$T = \begin{pmatrix} \frac{p}{2} + (1-p) & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} + (1-p) \end{pmatrix} = \begin{pmatrix} 1-\frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & 1-\frac{p}{2} \end{pmatrix}.$$

We obtained the binary symmetric channel, with $p' = p/2$.

b) *Maximize the mutual information over q to find the classical channel capacity of the depolarizing channel.*

The channel capacity of the binary symmetric channel, as has been shown in a previous exercise, is given by

$$C = 1 - H_{\text{bin}}(p/2), \quad H_{\text{bin}}(r) = -\left(r\log r + (1-r)\log(1-r)\right), \quad r \in [0,1].$$

c) *What happens to the channel capacity if we measure the final state in a different basis?*

Take an arbitrary basis $\left\{|\alpha\rangle, |\alpha^\perp\rangle\right\}$, where

$$|\alpha\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle, \quad |\alpha^\perp\rangle = \cos\left(\alpha + \frac{\pi}{2}\right)|0\rangle + \sin\left(\alpha + \frac{\pi}{2}\right)|1\rangle = -\sin\alpha|0\rangle + \cos\alpha|1\rangle.$$

Then

$$\begin{aligned} P_Y(\alpha) &= tr\left[|\alpha\rangle\langle\alpha|\ (\rho_X)\right] = tr\left[\begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}\begin{pmatrix} P_Y(0) & 0 \\ 0 & P_Y(1) \end{pmatrix}\right] \\ &= \cos^2(\alpha)P_Y(0) + \sin^2(\alpha)P_Y(1), \\ P_Y(\alpha^\perp) &= tr\left[|\alpha^\perp\rangle\langle\alpha^\perp|\ (\rho_X)\right] = tr\left[\begin{pmatrix} \sin^2\alpha & -\cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & \cos^2\alpha \end{pmatrix}\begin{pmatrix} P_Y(0) & 0 \\ 0 & P_Y(1) \end{pmatrix}\right] \\ &= \sin^2(\alpha)P_Y(0) + \cos^2(\alpha)P_Y(1). \end{aligned}$$

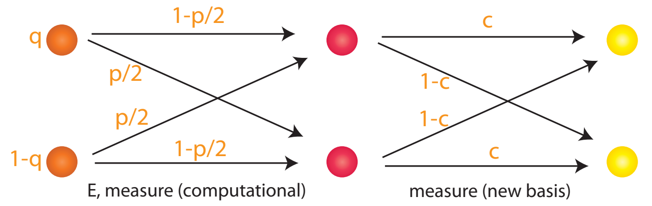

Figure 1: The result is a binary symmetric channel with $p' = 1 - c - p/2 + pc$.

We can see this result in the following way: take $c = \cos^2(\alpha)$. Then " preparing $q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$, applying E_p and measuring in basis $\{|\alpha\rangle, |\alpha^\perp\rangle\}$" is equivalent to the concatenation of two binary symmetric channels (Fig. 1).

The final probability distributions are the same if we apply E_p, measure in the computational basis, and then measure again in the new basis. This holds because E_p does not change the eigenbasis of the state, and is not necessarily true for a general TPCPM.

The capacity of the original channel is larger than the capacity of the concatenation of the two channels (because adding another channel just adds more noise, a fact otherwise known as the data processing inequality).

Exercise 6.2 A sufficient entanglement criterion

a) *Show that the* transpose *is a positive operation, and that it is basis-dependent.*

Let $\mathcal{H}$ be a Hilbert space and $\{|v_i\rangle\}_i$ a basis in $\mathcal{H}$. For any operator A on $\mathcal{H}$ we define the operation transpose $\mathcal{T}$ as

$$\mathcal{T} : A \mapsto A^T, \quad \mathcal{T}\left(\sum_{ij} a_{ij}\ |v_i\rangle\langle v_j|\right) \sum_{ij} a_{ji}\ |v_i\rangle\langle v_j|.$$

short hand: $\mathcal{T}(a_{ij}) = a_{ji}$

Since a matrix representation of an operator is in general basis-dependent, so is its transpose.

The positivity of $\mathcal{T}$ follows from

$$\mathcal{T}\,(UAU^*) = U^{*T} A^T U^T,$$

as can be seen from the definition. This means that if A has only positive eigenvalues, so does A^T, because unitaries do not change the eigenvalues, and the transpose of a unitary is unitary.

b) *Let $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a separable state, and let Λ_A be a positive operator on $\mathcal{H}_A$. Show that $\Lambda_A \otimes \mathbb{1}_B$ maps ρ on a positive operator.*

If $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a separable state, it can be written as convex combination of product states, $\rho = \sum_i p_A \sigma_A^i \otimes \sigma_B^i$, and

$$\Lambda_A \otimes \mathbb{1}_B \left(\sum_i p_A \sigma_A^i \otimes \sigma_B^i\right) = \sum_i p_A \Lambda_A(\sigma_A^i) \otimes \sigma_B^i.$$

All $\left\{\Lambda_A\left(\sigma_A^i\right)\right\}_i$ are positive operators. Since the set of positive operators is convex, we know that a convex combination of positive operators is still positive.

c) *Show that the* transpose *is a probable candidate by testing it on a Werner state (impure singlet)*

$$W = x|\psi^-\rangle\langle\psi^-| + (1-x)\,\mathbb{1}/4,$$

where $x \in [0,1]$ *and* $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$*. What happens to the eigenvalues of* W *when you apply transpose* $\otimes\, \mathbb{1}_B$*?*

W is only separable for $x < \frac{1}{3}$. On the other hand, W^t has three eigenvalues equal to $(1+x)/4$ and a fourth $(1-3x)/4$. This lowest eigenvalue is only positive if $x < \frac{1}{3}$. We conclude that a Werner state is separable only when its partial transpose is positive.

Remark: The Werner state is an example of an entangled state that nevertheless does not violate Bell's inequality. This means that in this sense the criterion we have constructed is stronger then Bell's inequality.

d) *Show that although the partial transpose is basis-dependent, the corresponding eigenvalues are independent under local basis transformations.*

Let $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an arbitrary density operator and ρ^t its partial transpose on A. If we perform a local change of basis then ρ transforms like

$$\rho \mapsto (U^{'} \otimes U^{''})\rho(U^{'} \otimes U^{''})^*,$$

and accordingly ρ^t,

$$\rho^t \mapsto (U^{'*T} \otimes U^{''})\rho^t(U^{'T} \otimes U^{''*}).$$

which is also a unitary transformation, thus leaving the eigenvalues of ρ^t unchanged.

Exercise 6.3 Uncertainty Principle

Let ρ *be a density operator,* A *and* B *observables, and* $t \in$R*. Show that*

1. $(A+itB)\rho(A+itB)^*$ *is positive (Alternatively, show directly that* $\langle (A+itB)(A+itB)^*\rangle$ *is positive)*

   Let ϕ be a normalised state from the corresponding Hilbert space. Then $\langle (A+itB)(A+itB)^*\phi, \phi\rangle = \langle (A+itB)\phi, (A+itB)\phi\rangle = |(A+itB)\phi|^2 \geq 0$

   Hence, we have that, $\langle (A+itB)(A+itB)^*\rangle = \langle A^2 + it(-AB+BA) + t^2B^2\rangle = \langle A^2\rangle - it\langle [A,B]\rangle + t^2\langle B^2\rangle \geq 0$

2. Show that $4\langle A^2\rangle_\rho \langle B^2\rangle_\rho \geq |\langle [A,B]\rangle_\rho|^2$

   Solving the obtained quadratic equation in t, and asking that determinant is negative, we obtain wanted inequality.

Hence deduce the Uncertainty Principle $\Delta_\rho[P]^2\Delta_\rho[X]^2 \geq \frac{1}{4}\hbar^2$.

Use the fact that $[P,X] = -i\hbar$, and the result 2.