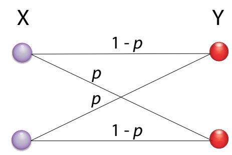
Exercise 5.1 Classical channels as trace-preserving completely positive maps.

In this exercise we will see how to represent classical channels as completely positive trace-preserving maps (CPTPMs).

a) Take the binary symmetric channel $\mathbf{p}$,



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Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1 - q$, we may express this as the 1-qubit mixed state $\rho_X = q\,|0\rangle\langle 0| + (1-q)\,|1\rangle\langle 1|$.

What is the quantum state ρ_Y that represents the final probability distribution P_Y in the computational basis?

b) Now we want to represent the channel as a map

$$\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) \mapsto \mathcal{S}(\mathcal{H}_Y)$$
$$\rho_X \to \rho_Y.$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \mathrm{Hom}(\mathcal{H}_X, \mathcal{H}_Y)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that

$$\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.$$

Find an operator-sum representation of $\mathcal{E}_{\mathbf{p}}$.

Hint: think of each operator $E_k = E_{xy}$ as the representation of the branch that maps input x to output y.

c) Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state ρ_X is not diagonal in the computational basis?

d) Now consider an arbitrary classical channel $\mathbf{p}$ from an n-bit space X to an m-bit space Y, defined by the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ as a map $\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ in the operator-sum representation.

Exercise 5.2 Different Quantum Channels

Consider two single-qubit Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a CPTP map

$$\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) \mapsto \mathcal{S}(\mathcal{H}_B)$$
$$\rho \to p\frac{\mathbb{1}}{2} + (1-p)\rho,$$

which is called *depolarizing channel*.

a) Find a Kraus representation for $\mathcal{E}_p$.

Hint: Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1, \quad \vec{r} \cdot \vec{\sigma} = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z, \tag{1}$$

where σ_x, σ_y and σ_z are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z).$$

b) What happens to the Bloch radius $\vec{r}$ of the initial state when we apply $\mathcal{E}_p$? How can this be interpreted?

c) Find Kraus representations for the following qubit channels

(i) The dephasing channel: $\rho \to \rho' = \mathcal{E}(\rho) = (1-p)\rho + p \,\mathrm{diag}(\rho_{00}, \rho_{11})$ (the off-diagonal elements are annihiliated with probability p).

(ii) The amplitude damping (damplitude) channel, defined by the action $|00\rangle \to |00\rangle$, $|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$.

Exercise 5.3 The Choi Isomorphism

The Choi Isomorphism can be used to determine whether a given mapping is a CPM. Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \;\mapsto\; (1-\alpha)\,\frac{\mathbb{1}_2}{2} + \alpha\,\big(\frac{\mathbb{1}_2}{2} + \sigma_x \rho \sigma_z + \sigma_z \rho \sigma_x\big), \qquad 0 \leq \alpha \leq 1, \tag{2}$$

where σ_i are Pauli matrices.

a) Use the Bloch representation to determine for what range of α these mappings are positive. What happens to the Bloch sphere?

b) Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:

$$\sigma_\alpha = (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{3}$$

For what range of α is the mapping a CPM?

c) Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.