Quantum Information Theory
Series 2

HS 14
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Exercise 2.1 Bloch sphere

In this exercise we will see how we may represent qubit states as points in a three-dimensional unit ball.
A qubit is a two level system, whose Hilbert space is equivalent to $\mathbb{C}^2$. The Pauli matrices together with the identity form a basis for 2×2 Hermitian matrices,

$$\mathcal{B} = \left\{ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{1}$$

where the matrices are represented in basis $\{|0\rangle, |1\rangle\}$.
We will see that density operators can always be expressed as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{2}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in an unit ball. The surface of that ball is usually known as the Bloch sphere.

a) Show that the Pauli matrices respect following commutation relations:

$$[\sigma_i, \sigma_j] := \sigma_i \sigma_j - \sigma_j \sigma_i = 2\varepsilon_{ijk}\sigma_k, \tag{3}$$

$$\{\sigma_i, \sigma_j\} := \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\mathbb{1}. \tag{4}$$

b) Show that the operator ρ defined in (2) is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:

1. Hermiticity: $\rho = \rho^\dagger$.
2. Positivity: $\rho \geq 0$.
3. Normalisation: $\mathrm{Tr}(\rho) = 1$.

c) Now do the converse: show that any two-level density operator may be written as (2).

d) Check that the surface of the ball is formed by all pure states.

e) Using (2) :

1. Find and draw the following vectors inside/on the Bloch sphere:
   - the fully mixed state
   - pure states that form the bases $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$.
     Hint: Use $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ *and* $|\circlearrowleft / \circlearrowright\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$.
2. Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. Do they represent pure or mixed states?

Exercise 2.2 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then:

$$H = \frac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \right) \tag{6}$$

a) Show that H is unitary.

b) What are the eigenvalues and eigenvectors of H?

c) What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?

d) Give a geometric interpretation of the action of H in terms of the Bloch sphere.