Phase Transitions and Critical Phenomena

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Exercise Sheet 10

HS 14
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Problem 1 Low temperature expansion

The partition function of the 2D Ising model at low temperatures can be written as

$$Z = \exp\left(-\frac{E_{\rm GS}}{T}\right)\left[1 + \sum_n \Delta Z^{(n)}\right] \tag{1}$$

where $E_{\rm GS}$ is the energy of the perfectly ordered state (the ground state) and $\Delta Z^{(n)}$ is the correction due to configurations that contain n flipped spins.

Let us define a small parameter

$$U = \exp\left(-\frac{2J}{T}\right) \tag{2}$$

where $2J$ is the excitation energy due to one broken bond. The first few corrections to the partition function can then be identified to be (cf. lecture notes)

$$\begin{aligned}
\Delta Z^{(1)} &= NU^4 \\
\Delta Z^{(2)} &= 2NU^6 + \frac{N(N-5)}{2}U^8 \\
\Delta Z^{(3)} &= 2NU^8 + 4NU^8 + 2N(N-8)U^{10} + \frac{N(N^2-15N+62)}{6}U^{12}. \\
&\dots
\end{aligned}$$

Summing up all contributions one finds that

$$Z = \exp\left(-\frac{E_{\rm GS}}{T}\right)\left[1 + NU^4 + 2NU^6 + \frac{N(N+9)}{2}U^8 + 2N(N+6)U^{10} + \dots\right] \tag{3}$$

Identify the missing higher $\Delta Z^{(n)}$ corrections necessary to reproduce formula (3) up to the U^8 order!