General relativity. Problem set 13.

HS 14 *Due:* Tue, December 16, 2014

1. Expanding or static spacetime?

Different properties of one and the same solution of the field equations may be more or less evident depending on the coordinates being used. An example is the de Sitter spacetime (no matter, $\Lambda = 3\alpha^2 > 0$), as illustrated by the following claim: The spatially flat, expanding solution

$$g = dt^2 - \alpha^{-2} e^{2\alpha t}\big((dy^1)^2 + (dy^2)^2 + (dy^3)^2\big) \tag{1}$$

(see (6.2) with $x^i \rightsquigarrow y^i$ and (6.6) with $a(t)$ as on p. 59) and the spherically symmetric, static solution

$$g = (1 - \alpha^2 r^2) dt^2 - (1 - \alpha^2 r^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{2}$$

(cf. eq. (P12.2) with $m = 0$) are actually the same spacetime! There is no contradiction in terms between an "expanding" and a "static" spacetime, since it may be viewed either way by different families of distinguished observers. Indeed, let u be their 4-velocity fields. Then "expanding" stands for free falling observers with u invariant under space symmetries; while "static" for (not necessarily free falling) observers with $u \parallel K$, a timelike Killing field.

By the following the above claim will be proved. Consider the pseudo-Riemannian manifold M ($\dim M = 4$) embedded in $\mathbb{R}^5$

$$-(x^0)^2 + \sum_{i=1}^{4} (x^i)^2 = \alpha^{-2} \tag{3}$$

with induced (Minkowski) metric

$$g = (dx^0)^2 - \sum_{i=1}^{4} (dx^i)^2 \ . \tag{4}$$

Remark: (M, g) is a solution of the field equations as one could show directly, but that will follow anyhow by (i) and (ii) below.

(i) Consider the coordinate change

$$x^0 = \alpha^{-1}(\operatorname{sh}\alpha t + r^2 e^{\alpha t}/2)\ , \qquad x^1 = \alpha^{-1}(\operatorname{ch}\alpha t - r^2 e^{\alpha t}/2)\ ,$$
$$x^2 = \alpha^{-1} e^{\alpha t} y^1\ , \qquad x^3 = \alpha^{-1} e^{\alpha t} y^2\ , \qquad x^4 = \alpha^{-1} e^{\alpha t} y^3$$

with $r^2 := \sum_{i=1}^{3} (y^i)^2$. Show that (t, y^1, y^2, y^3) are coordinates for M or part of it. Use them to express the metric (4).

(ii) Show the same as in (i) for the coordinates (t, r, θ, φ) and the transformation

$$x^0 = \alpha^{-1}\sqrt{1 - \alpha^2 r^2}\,\operatorname{sh}\alpha t\ , \qquad x^1 = \alpha^{-1}\sqrt{1 - \alpha^2 r^2}\,\operatorname{ch}\alpha t\ ,$$
$$x^2 = r\sin\theta\cos\varphi\ , \qquad x^3 = r\sin\theta\sin\varphi\ , \qquad x^4 = r\cos\theta\ .$$

(iii) Determine the worldlines of the distinguished observers for the metrics (1) and (2) as families of curves on the one-sheeted hyperboloid (3). Visualization in $\mathbb{R}^3$ is possible by restricting to $x^3 = x^4 = 0$.

2. Radial free fall

Find the motion $r(\tau)$ of a particle falling radially inward from $r = R$ towards a black hole and starting from rest in Schwarzschild coordinates. Note that $r(\tau)$ can not be expressed in closed form, but there is a parametric representation $r = r(\eta)$, $\tau = \tau(\eta)$ which can.

Hint: The radial equation has been encountered before in another context.