General relativity. Problem set 9.

HS 14 *Due:* Tue, November 18, 2014

1. Energy conditions

In the following a reference frame means a local basis e_α ($\alpha = 0, 1, 2, 3$) with $(e_\alpha, e_\beta) = \eta_{\alpha\beta}$. The 4-velocity of an observer at rest therein is $u = e_0$.

i) The 4-momentum $(p^\mu)_{\mu=0}^3 = (E/c, \vec{p})$ of a particle of mass $m \geq 0$ satisfies $E \geq 0$ in *every* reference frame. This is formulated in general covariant form as: p^μ is timelike or lightlike, and future oriented. The generalization on the energy-momentum tensor is: $T^{00} \geq 0$ in *every* reference frame. Formulate this **weak energy condition** in a covariant way.

Hint: The 4-velocity of an observer is timelike.

ii) The **strong energy condition** says $T^{00} + \sum_{i=1}^3 T^{ii} \geq 0$ in *every* reference frame. Formulate also this condition in a covariant way. Show: by use of the Einstein field equations this means that free falling matter attracts itself. More precisely: embed a reference geodesic $x(\tau)$, with initial 4-velocity $(dx/d\tau)|_{\tau=0} = e_0$, in a 1-parameter family of geodesics $x(\tau, \lambda)$, all with initial 4-velocity e_0. Consider the separation $n_a(\tau) = (dx/d\lambda)|_{\lambda=0}$ between close geodesics with initial condition $n_a(0) = e_a$, ($a = 1, 2$ or 3). Attraction means (averaging over the directions), cf. (4.23), that

$$-\mathrm{Ric}(u, u) \leq 0 \, .$$

Remark: The strong energy condition does not imply the weak one.

iii) The **dominant energy condition** is a strenghtening of the weak energy condition. It requires the energy flow $(T^{\mu 0})_{\mu=0}^3$ to be timelike or lightlike, and future oriented ($T^{00} \geq 0$), in *every* reference frame. (That means that the propagation velocity of the energy is $\leq c$). Formulate this condition in a covariant way and show that it is equivalent to $T^{00} \geq |T^{\alpha\beta}|$ for every α, β in *every* reference frame (hence the name of the condition).

iv) What do the conditions (i-iii) imply for the perfect fluid, the electromagnetic field, and for the vacuum (cosmological term with $\Lambda > 0$)?

Hint: It is easier to use the conditions (i-iii) in the above form, i.e. in terms of tensor components, than in the covariant form.

2. On Hawking's singularity theorem

Let (M, g) be a pseudo-Riemannian manifold of signature $(+, -, -, -)$ and $\Sigma \subset M$ be a spacelike 3-surface without boundary having normal u: $g(u, u) = 1$, $g(u, X) = 0$ for any vector field X on Σ. Conventionally call the side of Σ distinguished by u its future side. Define a (symmetric) tensor K of type $\binom{0}{2}$ on Σ, called extrinsic curvature, by

$$K(X, Y) = -g(\nabla_X u, Y) \, ,$$

where X, Y are vector fields on Σ.

Theorem (roughly). Suppose Σ is compact and $\operatorname{tr} K \leq C < 0$ on Σ. Suppose

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{1}$$

for any timelike vector field ξ on M. Then there is a timelike geodesic starting from the past side of Σ, which ends in a singularity of M. In fact, it reaches it within proper time $\leq 3/|C|$.

i) For g a solution of the field equations (5.11), eq. (1) amounts to which energy condition?

ii) Let Σ be a time slice in a Friedmann model (it is compact for $k = +1$) and let $u = (1, 0, 0, 0)$ in chart A on p. 50. Show that if $\dot{a}(t_0) > 0$ (expansion) at some time $t_0 > 0$, then there is a singularity in the past of Σ. How far back in proper time can it at most occur?

Remark: That singularity is of course the Big Bang implied by the Friedmann equations, at least for $\Lambda = 0$. The theorem however shows that homogeneity and isotropy are not essential.