General relativity. Problem set 6.

HS 14 *Due:* Tue, October 28, 2014

1. A free fall

A clock C_2 lies at fixed height h above a clock C_1 in a reference frame in which the gravitational field g is constant. Two masses are dropped in short succession from C_2 and in a time interval Δt_2 as measured by the local clock. What is the time difference Δt_1 measured by C_1 between their arrival times?

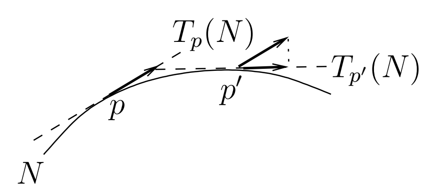
Hint: Use the equivalence principle (hence making use of a local inertial frame) and special relativity. The time difference Δt_2 is much shorter than the time of fall.

2. Newton's equation as a geodesic equation

Rewrite Newton's equation of motion $\ddot{\vec{x}} = -\vec{\nabla}\varphi$ in a gravitational potential $\varphi(\vec{x})$ as a geodesic equation for $(t, \vec{x}(t))$ in a 4-dimensional spacetime. Identify the Christoffel symbols of the affine connection. Show that the latter is (i) symmetric but (ii) cannot be metric.

Hint for (ii): Show that $R^i{}_{0k0} = \varphi_{,ik}$, $R^i{}_{jk0} = 0$, $(i, j, k \neq 0)$. Assuming the connection to be determined by a metric g, compute R_{i0j0} and R_{0ij0}, and obtain a contradiction to a symmetry of the Riemann tensor.

3. On the Levi-Civita connection



A special case and an illustration of the result discussed below is as follows. Consider a surface N, e.g. the sphere, embedded in the Euclidean space $\mathbb{R}^3$. A curve in N is also one in the ambient space $\mathbb{R}^3$, but parallel transport depends on how the curve is viewed. In fact, in the first case a vector in $T_p(N)$ is transported from p to p' in the Euclidean sense, followed by the projection to $T_{p'}(N)$; more precisely, this is so infinitesimally, i.e. in the limit $p' \to p$.

Let $N \subset M$ be a submanifold of the manifold M with metric g. It naturally carries the induced metric obtained by restricting the map $(X, Y) \mapsto g(X, Y)$ to vector fields X, Y on N. At any point $p \in N$ let $P_p : T_p(M) \to T_p(N)$ be the orthogonal projection associated to g_p, i.e.

$$g_p(X, Y) = g_p(P_p X, Y) ,$$

where $X \in T_p(M)$ and $Y \in T_p(N)$. The metric g determines a Levi-Civita connection $\nabla^{(M)}$ on M and, through its induced metric, one on N too, $\nabla^{(N)}$. Show that

$$(\nabla^{(N)}_X Y)_p = P_p(\nabla^{(M)}_X Y)_p \tag{1}$$

for any vector fields X, Y on N.

Hint: Note that $\nabla^{(M)}_X Y$ is well-defined for vector fields on N. Show that $P \circ \nabla^{(M)}$ has the properties of $\nabla^{(N)}$.