# General relativity. Problem set 5.

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*HS 14* *Due:* Tue, October 21, 2014

## 1. Torsion and Hessian

The Hessian of a (smooth) function $f\colon \mathbb{R}^n \to \mathbb{R}$ is the matrix $\partial^2 f = (f_{,ij})_{i,j=1}^n$ of its second derivatives $f_{,ij} = f_{,ij}(x)$. It is evidently symmetric.

For a function on a manifold $M$, the gradient covector $df$ has the first derivatives $(f_{,i})_{i=1}^n$ as its components with respect to a coordinate basis; however the second derivatives do not transform as tensor components.

Given an affine connection $\nabla$, the gradient is $df = \nabla f$ (why?) and a substitute Hessian may be defined as

$$H = \nabla^2 f \equiv \nabla\nabla f \,.$$

It is a tensor field of type $\binom{0}{2}$. Show: It is symmetric, $H(X,Y) = H(Y,X)$ for all $f \in \mathcal{F}$, iff the torsion vanishes.

## 2. Euclidean metric in polar coordinates

Consider the Euclidean plane as a Riemannian manifold $M = \mathbb{R}^2 \ni (x^1, x^2) = x$ with metric $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$. Compute the metric in polar coordinates $r, \varphi$ and the Christoffel symbols (3.6) of the Levi-Civita connection. Verify that they agree with those computed in Problem 3.1.

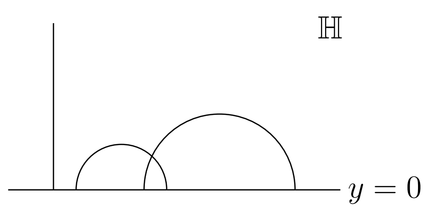
## 3. Geodesics in the hyperbolic plane

Consider the hyperbolic plane: $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ with the metric $g = y^{-2}(dx \otimes dx + dy \otimes dy)$.

a) Write the geodesic equation.

b) Find the quantities which are conserved along the geodesics.

c) Show that the geodesics are the Euclidean half-circles (including half-lines), centered on the line $y = 0$.




*Hint:* Show that the (extrinsic) curvature

$$\rho = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

is constant along geodesics.

**4. An affine connection on Lie groups**

Consider a Lie group $G$ (see Problem 3.2).

a) Show that there is a unique affine connection $\nabla$ on $G$ with the properties that

i) for any left-invariant vector field $V$ on $G$, the tangent vectors $d\gamma/dt = V_{\gamma(t)}$ to any of its orbits $\gamma(t)$ are parallel transported along it;

ii) the torsion vanishes.

To define a connection $\nabla$ is tantamount to prescribing its coefficients $\langle e^\alpha, \nabla_{e_\beta} e_\gamma \rangle$, see (2.12), w.r.t. vectors $e_\alpha$, resp. covectors $e^\beta$ forming dual bases $(e_1, \ldots, e_n)$, $(e^1, \ldots, e^n)$. These fields are not necessarily coordinate bases, see (2.17, 2.18).

b) Show that the connection of part (a) has coefficients

$$\langle e^\gamma, \nabla_{e_\alpha} e_\beta \rangle = \frac{1}{2} {C^\gamma}_{\alpha\beta} , \tag{1}$$

where the $e_\alpha$ are left-invariant basis fields and ${C^\gamma}_{\alpha\beta} = -{C^\gamma}_{\beta\alpha}$ their structure constants (see Problem 3.2 iv).

*Hint:* What are the equations for $\nabla$ expressing (i, ii)? Rewrite them in terms of a left-invariant basis $e_i$ and obtain two conditions on $\nabla_{e_\alpha} e_\beta$. Finally, take their sum and difference.