General relativity. Problem set 3.

HS 14 *Due:* Tue, October 7, 2014

1. Parallel transport in polar coordinates

Consider the Euclidean plane $\mathbb{R}^2 \ni (x^1, x^2) = x$ as a manifold with chart: $\mathrm{id} : x \mapsto x$. Define a Cartesian parallel transport $T_x(\mathbb{R}^2) \ni v \mapsto v' \in T_{x'}(\mathbb{R}^2)$ along any curve by requiring that v and v' have the same components. Compute the Christoffel symbols of this parallel transport in polar coordinates r, φ.

Hint: What is the transformation from Cartesian to polar components, $(v^1, v^2) \mapsto (v^r, v^\varphi)$, of a vector v?

2. Lie groups and Lie brackets

Consider the group of regular, real $n \times n$ matrices:

$$\mathrm{GL}(n, \mathbb{R}) = \{m = (m_{ij})_{i,j=1}^n \mid m_{ij} \in \mathbb{R}\,, \det m \neq 0\}$$

equipped with matrix multiplication. The unit element is $e = (\delta_{ij})_{i,j=1}^n$. $\mathrm{GL}(n, \mathbb{R})$ is a differentiable manifold of dimension n^2.

The tangent space at e consists of tangents $\dot{m}(0)$ to curves $m(t)$ with $m(0) = e$ and is denoted by

$$\mathfrak{gl}(n, \mathbb{R}) = T_e(\mathrm{GL}(n, \mathbb{R})) = \{x = (x_{ij})_{i,j=1}^n \mid x_{ij} \in \mathbb{R}\}\ .$$

A (matrix) Lie group G is a subgroup and a submanifold of $\mathrm{GL}(n, \mathbb{R})$. Examples (besides of the trivial $G = \mathrm{GL}(n, \mathbb{R})$) are

a) the orthogonal group

$$\mathrm{O}(n) = \{r \in \mathrm{GL}(n, \mathbb{R}) \mid r^T r = e\}\ ,$$

where r^T is the transpose of r;

b) the Lorentz group

$$\mathrm{SO}(1, 3) = \{l \in \mathrm{GL}(4, \mathbb{R}) \mid l^T \eta l = \eta\}\ ,$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$.

The tangent space at $e \in G$ consists of matrices:

$$\mathrm{Lie}(G) := T_e(G) \subset \mathfrak{gl}(n, \mathbb{R})\ .$$

i) Find $\mathrm{Lie}(G)$ for G in the examples a), b).

ii) Show that for any $x_1, x_2 \in \mathrm{Lie}(G)$

$$\alpha_1 x_1 + \alpha_2 x_2 \in \mathrm{Lie}(G)\ , \quad (\alpha_1, \alpha_2 \in \mathbb{R})\ , \tag{1}$$

$$[x_1, x_2] := x_1 x_2 - x_2 x_1 \in \mathrm{Lie}(G)\ , \tag{2}$$

moreover,

$$[\alpha_1 x_1 + \alpha_2 x_2, x] = \alpha_1 [x_1, x] + \alpha_2 [x_2, x] \ . \tag{3}$$

Hint: If $m_1(t), m_2(t) \in G$ are curves through e, so are $m_i(\lambda_i t)$, $(i = 1, 2)$, $m_1(t)m_2(t)$ and, for any s, $m_1(t)m_2(s)m_1(t)^{-1}m_2(s)^{-1}$.

A linear space equipped with a bilinear, antisymmetric bracket $[\cdot, \cdot]$ satisfying the Jacobi identity is called a Lie algebra. Examples: $\mathrm{Vf}(M)$, the vector fields on M (see lecture notes); $\mathrm{Lie}(G)$ is the Lie algebra of G (see above).

For any $g \in G$, let λ_g be the left-multiplication on G:

$$\lambda_g : G \longrightarrow G \ , \quad h \longmapsto gh \ .$$

It is a diffeomorphism. Among the vector fields X on G, consider those which are left-invariant, meaning

$$(\lambda_g)_* X = X \ , \qquad (g \in G) \ ; \tag{4}$$

equivalently, $(\lambda_g)_* X_h = X_{gh}$. Clearly, they form a linear space. Show that

iii) They also form a Lie algebra w.r.t. the Lie bracket of $\mathrm{Vf}(G)$. *Hint:* Quite generally, $\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]$ (see notes, p. 7).

iv) If the vector fields are left-invariant, then the functions ${C^\gamma}_{\alpha\beta}$ in Eq. (P2.1) are constants (called structure constants). How does Eq. (P2.4) simplify? *Hint:* $\varphi_*(fX) = (f \circ \varphi^{-1})\varphi_* X$.

v) The left-invariant vector fields are in bijective relation to the tangent vectors at e

$$X \ \longleftrightarrow \ x \in T_e(G) = \mathrm{Lie}(G)$$

such that $X_e = x$.

vi) The bijection is a Lie algebra isomorphism:

$$\begin{aligned} \alpha_1 X_1 + \alpha_2 X_2 \ &\longleftrightarrow \ \alpha_1 x_1 + \alpha_2 x_2 \ , \\ [X_1, X_2] \ &\longleftrightarrow \ [x_1, x_2] \ . \end{aligned}$$

Hint: All functions $f \in \mathcal{F}(G)$ are obtained by restriction to $G \subset \mathrm{GL}(n, \mathbb{R})$ from $\tilde{f} \in \mathcal{F}(\mathrm{GL}(n, \mathbb{R}))$,

$$f(g) = \tilde{f}(m)|_{m=g} \ , \qquad (g \in G) \ .$$

Then $X_e = x$ states

$$X_e f = x_{ij} \frac{\partial \tilde{f}}{\partial m_{ij}}\Big|_{m=e} \ . \tag{5}$$