General relativity. Problem set 1.

HS 14 *Due:* Tue, September 23, 2014

1. The sphere as a manifold

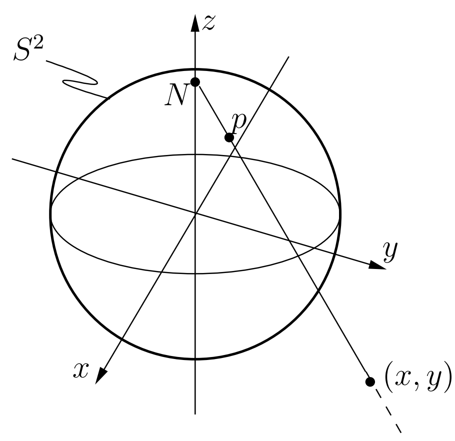
Consider the sphere

$$S^2 = \{p = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

and its covering $S^2 = U_+ \cup U_-$ by the two open sets

$$U_\pm = S^2 \setminus \{(0, 0, \pm 1)\}$$

obtained by removing the north, resp. south pole from S^2. The stereographic projection $p \mapsto (x, y)$ shown in the figure provides a chart for U_+ with coordinate neighborhood $\mathbb{R}^2 \ni (x, y)$; similarly there is one for U_- with neighborhood $\mathbb{R}^2 \ni (\bar{x}, \bar{y})$. On which subset of $\mathbb{R}^2$ is the transition function $(x, y) \mapsto (\bar{x}, \bar{y})$ defined? Compute that function.



2. Tensors

a) Show that not all tensors in

$$\begin{aligned} V \otimes V &= \{T \mid T \text{ is a bilinear form over } V^* \times V^*\} \\ &= \{\text{linear combinations of tensors } v_1 \otimes v_2 \mid v_1, v_2 \in V\} \end{aligned}$$

are of the form $v_1 \otimes v_2$ (simple tensors).

b) Identify $V \otimes W^*$ with the linear space $\mathcal{L}(W, V)$ of linear maps $W \to V$.