Exercise 1. ***Non-linear sigma model***

The non-linear sigma model with an $SO(4)$ symmetry can be obtained from the linear one by introducing a field dependent rotation.

$$\phi(x) = R_{n4}(x)\sigma(x).$$

The so obtained Lagrangian (see your Lecture notes or WB. 19.5.) is then given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + 2\sigma^2\vec{D}_\mu\vec{D}^\mu - \frac{1}{2}m^2\sigma^2 - \frac{1}{4}\lambda\sigma^4,$$

with

$$\vec{D}_\mu = \frac{\partial_\mu\vec{\xi}}{1+\vec{\xi}^2}$$

(a) Work out the infinitesimal transformation behaviour of the $\sigma, \vec{\xi}$ and $\vec{D}_\mu$ under the $SO(4)$ symmetry. Show that they are indeed non-linear. Hint: Separate the transformations affecting the 4 component from the rest.

Exercise 2. ***The Higgs and the linear sigma model***

In this exercise we want to make contact between the linear sigma model and the Higgs of the Standard Model before electro-weak symmetry breaking. To this end we introduce a complex doublet H of $SU_L(2)$ and with $Y = \frac{1}{2}$ hyper-charge. We use this doublet to define the matrix

$$\phi = (H,\ H^c), \qquad H^c = -i\sigma^2 H^*.$$

(a) Show that ϕ satisfies the pseudo-reality condition $\phi = -i\sigma^2\phi^*\sigma^2$.

(b) Show that ϕ can be written in terms of 4 real scalar fields

$$\phi = h_4 I + i\sum_{i=1}^{3} h_i\sigma^i,$$

where the σ_i are the Pauli matrices. Identify the components h_n.

(c) Show that the Higgs Lagrangian coincides with the linear sigma model Lagrangian.

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu H^\dagger\partial^\mu H - \frac{1}{2}H^\dagger H - \frac{\lambda}{4}\left(H^\dagger H\right)^2$$

(d) The symmetry group of the Higgs Lagrangian is actually larger than $SU_L(2)\times U(1)_Y$. Which one is it?

(e) Work out how the generators of $SU_L(2)\times U(1)_Y$ act on the Higgs doublet H and on the linear sigma model fields h_n.