Exercise 1. Show that the $SU_L(2) \otimes SU_R(2)$ group has the same algebra as $SO(4)$. You can proceed as follows: Compute explicitly the 6 generators of $SO(4)$ in terms of 4×4 matrices. Show that you can construct the same commutation relations with the $SO(4)$ generators as with the $SU_L(2) \otimes SU_R(2)$ generators.

Exercise 2. Consider the $SO(4)$ invariant linear sigma model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_n \partial^\mu \phi_n - \frac{1}{2}m^2|\vec{\phi}|^2 - \frac{\lambda}{4!}\left(|\vec{\phi}|^2\right)^2.$$

Show that the space-time dependent reparametrisation

$$\begin{aligned} \phi_n &= R_{n4}\sigma, \\ R_{44} &= \frac{1-\vec{\xi}^2}{1+\vec{\xi}^2}, \qquad R_{a,4} = \frac{2\xi^a}{1+\vec{\xi}^2}, \qquad a \in \{1,2,3\} \end{aligned}$$

yields the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + 2\sigma^2 \vec{D}_\mu \vec{D}^\mu - \frac{m^2}{2}\sigma^2 - \frac{\lambda}{4!}\sigma^4, \qquad \vec{D}_\mu = \frac{\partial_\mu \vec{\xi}}{1+\vec{\xi}^2}.$$

Exercise 3. The charges associated with the chiral symmetry of the 2-flavour QCD Lagrangian are

$$\begin{aligned} \vec{T} &= i\int d^3x \bar{q}\vec{t}\gamma^0 q, \\ \vec{X} &= i\int d^3x \bar{q}\vec{t}\gamma^0\gamma_5 q. \end{aligned} \tag{1}$$

Consider the chiral four-vectors $\Phi^\pm_\mu$

$$\begin{aligned} \vec{\Phi}^+ &= i\bar{q}\gamma_5\vec{t}q, \qquad \Phi^+_4 = \frac{1}{2}\bar{q}q, \\ \vec{\Phi}^- &= i\bar{q}\vec{t}q, \qquad \Phi^-_4 = -\frac{1}{2}\bar{q}\gamma_5 q. \end{aligned} \tag{2}$$

Check the following commutation relations:

$$\begin{aligned} \left[T^i, T^j\right] &= i\epsilon^{ijk}T^k, \\ \left[T^i, X^j\right] &= i\epsilon^{ijk}X^k, \\ \left[X^i, X^j\right] &= i\epsilon^{ijk}T^k, \\ \left[T^i, \Phi^\pm_n\right] &= -\mathcal{T}^i_{nm}\Phi^\pm_m, \\ \left[X^i, \Phi^\pm_n\right] &= -\mathcal{X}^i_{nm}\Phi^\pm_m, \end{aligned} \tag{3}$$

where the indices n, m run from 1 to 4 and i, j, k run from 1 to 3. Find that

$$\mathcal{T}^i_{jk} = -i\epsilon^i_{jk}, \qquad \mathcal{X}^i_{j4} = -\mathcal{X}^i_{4j} = -i\delta^i_j, \tag{4}$$

and all other components are zero.