Exercise 1. ***Goldstone's theorem and the effective potential***

Let us consider ϕ^4 theory for a scalar doublet $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ with the action

$$S[\phi_i] = \int \mathrm{d}^4 p \, \mathcal{L}[\phi_i], \qquad \text{with} \quad \mathcal{L}[\phi_i] = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi^i - \frac{1}{2} m^2 \phi_i^2 - \frac{\lambda}{4!}(\phi_i^2)^2,$$

where summation over i is implied.

(a) What is the global symmetry of this action? Give an example of a generator and an explicit symmetry transformation.

The effective action $i\Gamma[\phi]$ has been introduced in the QFT II as the sum of all one-particle irreducible (1PI) connected graphs, with arbitrary number of external lines such that each external lines comes with a factor of ϕ. We want to analyse a space-time independent field $\vec{\phi}_i^0$ and compute the contributions to the effective action arising from quantum fluctuations. The effective action for the constant field can be written using a shifted action $S[\phi + \phi_0]$ as

$$e^{i\Gamma[\phi_i^0]} = \int_{\substack{\text{1PI} \\ \text{connected}}} \mathcal{D}\vec{\phi}(x) \, \mathrm{e}^{iS[\phi_i + \phi_i^0]} \, .$$

We define the effective potential $V[\phi_i^0]$ via

$$\Gamma[\phi_i^0] = -\mathcal{V} V(\phi_i^0) \qquad \mathcal{V} = \int d^d x = (2\pi)^d \delta^d(p-p).$$

(b) Compute the shifted action $S[\phi_i + \phi_i^0]$ and rewrite it in terms of the field-dependent masses $\mu_i^2(\phi_i^0)$. What are the new vertices? Do they all contribute to the 1PI diagrams? Show that the zero-loop contribution to the effective potential is simply given by

$$V^{(0)}(\phi_i^0) = \frac{1}{2} m^2 (\phi_i^0)^2 + \frac{\lambda}{4!}((\phi_i^0)^2)^2.$$

Draw all diagrams contributing to the one- and two-loop corrections to the effective potential.

(c) The one-loop contribution for a single scalar field to the effective action is given by the following path integral

$$\begin{aligned} i\Gamma^{(1)}[\phi_0] &= \log \int \mathcal{D}\phi(x) \, \exp\left\{ \frac{i}{2} \int \mathrm{d}^4 x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\mu^2(\phi_0)\phi^2 \right\} \\ &= \log \det[\frac{-i}{\pi}(\partial^2 + \mu^2 - i\epsilon)]^{-1/2} = -\frac{1}{2} \mathrm{Tr} \, \log \frac{-i}{\pi}(\partial^2 + \mu^2 - i\epsilon) \end{aligned}$$

where the operator trace can be computed in momentum space

$$\mathrm{Tr} \, \log \frac{-i}{\pi}(\partial^2 + \mu^2 - i\epsilon) = \int \mathrm{d}^4 p \, K_{p,p}$$

with

$$K_{p,q} = \int \frac{\mathrm{d}^4x}{(2\pi)^2} \int \frac{\mathrm{d}^4y}{(2\pi)^2} \mathrm{e}^{-ixp+iyq} \log\left(\frac{i}{\pi}(\partial_x \cdot \partial_y - \mu^2 + i\epsilon)\right) \delta^4(x-y).$$

Using this expression, show that the one-loop contribution to the effective potential is given by

$$V^{(1)}(\phi_0) = I(\mu^2(\phi_0)) = -\frac{i}{2(2\pi)^4} \int \mathrm{d}^4p \log \frac{i}{\pi}(p^2 - \mu^2(\phi_0) + i\epsilon).$$

What happens in the case of our two scalar fields? Calculate the effective potential in this case.

(d) Expand the effective potential up to $\mathcal{O}(g^2)$ in the coupling constant, neglect all terms not proportional to the fields and show that the one-loop contribution is given by

$$V^{(1)}(\phi_i^0) = I(m^2) = \frac{ig}{3(2\pi)^4}(\phi_i^0)^2 \int \frac{d^4p}{p^2 - m^2 + i\epsilon} + \mathcal{O}(g^2).$$

(e) The integral $I(m^2)$ is badly UV divergent and hence needs to be regularized. This can be done using the following trick: First differentiate $I(m^2)$ two times with respect to m^2, then integrate over p after performing a Wick rotation ($m^2 > 0$), and finally integrate back two times with respect to m^2. Using this, show that $I(m^2)$ can be written as

$$I(m^2) = \frac{g}{3(4\pi)^2} m^2 (\phi_i^0)^2 \log(m^2) + gA(\phi_i^0)^2 m^2 + gB(\phi_i^0)^2,$$

where A and B are unknown integration constants. Note that these constants must be infinite, since $I(\mu^2)$ and $I'(\mu^2)$ are divergent. Renormalize the mass and coupling using

$$m_R^2 = m^2 + 2gB + 2m^2 gA + \mathcal{O}(g^2), \qquad g_R = g + \mathcal{O}(g^2).$$

What happens with the mass of the scalar fields?

(f) We can take a look back at our calculation and try to understand the situation in the case where $m^2 < 0$. Our Wick rotation in the computation of the integral $I(m^2)$ is not valid anymore. Compute the effective potential up top $\mathcal{O}(g^2)$ by Wick rotating p_i, $i = 1, 2, 3$ instead of p_0. Plot $V(\vec{\phi}_0)$ as a function of $(\phi_i^0)^2$ and see that it develops a non-trivial minimum in the case $m_R^2 < 0$, $g_R > 0$. Find a value of the fields that minimise the zero- and one-loop effective potential., i.e. the vacuum.

(g) Recall that the second derivative of the effective potential with respect to the fields is related to the reciprocal of the momentum space propagator. At the vacuum, i.e. at zero momentum, this corresponds to the mass matrix of the theory

$$\frac{\partial^2 V(\vec{\phi})}{\partial\phi_i \partial\phi_j}\bigg|_{\phi=\phi_0} = M_{ij}$$

Show for the zero- and one-loop effective potential that the mass matrix has two distinct eigenvalues for $m_R^2 < 0$. Show that one of the eigenvalues is zero in both cases. What conclusions can you draw?