Introduction to String Theory

Problem Sets

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Contents

1.1. On the importance of quantum gravity

Let us get some intuition on the order of magnitudes:

a) Consider a gravitational atom, an electron bound to a neutron by the gravitational force. Electromagnetic dipole effects can be neglected. Perform a semiclassical calculation to determine the radius of the orbit of the electron (first Bohr radius). Relate this radius to an appropriate distance in physics.

b) In "natural units", where $\hbar$, G and c are set to 1, a stellar black hole radiates like a black body at a temperature given by $kT = 1/8\pi M$. Give the temperature in SI units (reinsert G, $\hbar$ and c) and calculate the temperature of a black hole weighing one solar mass.

1.2. Relativistic point particle

The action of a relativistic point particle is given by

$$S_{\text{rp}} = -\alpha \int_{\mathcal{P}} ds$$

with the relativistic line element

$$ds^2 = -\eta_{\mu\nu} dX^\mu dX^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and α a (yet to be determined) constant. The path $\mathcal{P}$ between two points X_1^μ and X_2^μ can be parametrised by a parameter τ. The integral over the line element ds becomes an integral over the parameter

$$S_{\text{rp}} = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} \,. \tag{1.1}$$

a) Parametrise the path by the time coordinate t and take the non-relativistic limit $|\dot{\vec{x}}| \ll c$ to determine the value of the constant α. Characterise the appearing terms.

b) Derive the equations of motion by varying the action in (1.1). (You may set $c = 1$ from now on.) *Hint:* Calculate the canonically conjugate momentum P_μ first.

c) Show that the form of the action is invariant under reparametrisations $\tau' = f(\tau)$. This is what we call *manifestly* invariant.

d) Consider an electrically charged particle with charge q. In the presence of an external gauge field A_μ there is an additional term in the action governing the interaction between particle and field given by

$$S_{\text{em}} = \frac{q}{c} \int d\tau \, A_\mu(X) \, \frac{\partial X^\mu}{\partial \tau} \,.$$

Find the variation of $A^\mu(X)$ under a variation of the path δX^μ. Vary the action $S = S_{\text{rp}} + S_{\text{em}}$ w.r.t. X^μ to find the equations of motion for the particle. *Hint:* Use P_μ from above to simplify the expression.

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1.3. Polynomial action

There is another way to write the action of a relativistic particle. We introduce an auxiliary field called vierbein (or "einbein" in this case) e along the worldline of the particle and rewrite the action in the form

$$S_{\mathrm{pp}} = \int d\tau (e^{-1}\dot{X}^2 - m^2 e).$$

a) Show that S_{pp} is equivalent to S_{rp} above by eliminating the einbein from the action.

b) Derive the equations of motion by varying S_{pp} with respect to X and e.

c) Show that S_{pp} is invariant under infinitesimal reparametrisations $\delta\tau = -\epsilon(\tau)$ to linear order in ϵ. First find the correct transformation of X^μ. The einbein transforms like (can you derive it?)

$$\delta e = \partial_\tau(\epsilon(\tau)e).$$

d) Reparametrisation invariance is a gauge invariance. Thus by fixing a gauge we can eliminate one degree of freedom. Assume a gauge in which e is constant. Show that e can be written like

$$e = \frac{\ell}{\tau_2 - \tau_1},$$

where ℓ is the invariant length of the worldline for a path starting at $X^\mu(\tau_1)$ and ending at $X^\mu(\tau_2)$. *Hint:* Meditate on the role of the einbein and on how to define ℓ.

2.1. Symmetries of the classical string

In this exercise we examine the classical symmetries of the Polyakov string

$$S_P = -\frac{T}{2}\int d^2\xi\ \sqrt{-g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}.$$

We start with the global symmetries – Lorentz and translational symmetry – and proceed to gauge symmetries – reparametrisation and Weyl symmetry.

a) Consider the transformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\mu + a^\mu$$

which is a combination of a Lorentz transformation and a translation, a.k.a. a Poincaré transformation. Using the Noether procedure show that in conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ the Noether currents corresponding to these symmetries are given by

$$\mathcal{P}^\alpha_\mu = -T\partial^\alpha X_\mu, \qquad \mathcal{J}^\alpha_{\mu\nu} = \mathcal{P}^\alpha_\mu X_\nu - \mathcal{P}^\alpha_\nu X_\mu.$$

b) Find and identify the conserved charges associated with Lorentz boosts and time translations. *Hint:* For the Lorentz boost assume $X^0 = t$.

c) Show that the Polyakov string action is invariant under a reparametrisation $\xi^\alpha \to \tilde{\xi}^\alpha(\xi)$.

d) Show that the Polyakov string is also invariant under Weyl transformations: local length-changing but angle preserving transformations of the metric $g_{\alpha\beta} \to e^{2\omega(\xi)} g_{\alpha\beta}$.

e) Consider an infinitesimal Weyl transformation

$$\delta g_{\alpha\beta} = 2\omega g_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0$$

and show that Weyl symmetry implies the vanishing of the trace of the worldsheet energy-momentum tensor

$$T^\alpha{}_\alpha = 0.$$

Hint: The variation of the determinant is given by

$$\delta \det g = -\det g\, g_{\alpha\beta}\,\delta g^{\alpha\beta}.$$

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2.2. Classical spinning strings

The classical solution for the wave equation is given by

$$X^\mu(\sigma,\tau) = X^\mu_{\mathrm{L}}(\tau+\sigma) + X^\mu_{\mathrm{R}}(\tau-\sigma),$$

the constraints by

$$\dot{X}\cdot X' = 0 \qquad \text{and} \qquad \dot{X}^2 + X'^2 = 0.$$

It will be beneficial to work in static gauge $X^0(\sigma,\tau) = R\tau$ (τ being the worldsheet time).

a) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R\cos(\sigma)\cos(\tau) \\ X^2 &= R\cos(\sigma)\sin(\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation and that it fulfils the constraints. Calculate the energy $P^0 = E$ and the angular momentum J_{ij} of the solution.

b) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R\cos(\sigma)\cos(\tau) \\ X^2 &= R\cos(2\sigma)\sin(2\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation but does not fulfil the constraint equations.

c) Closed strings can develop cusps. These points σ_0 on the string are indicated by a singularity in the parametrisation

$$\frac{\partial\vec{X}}{\partial\sigma}(\sigma_0,t) = 0.$$

Show that the string reaches the speed of light at a cusp. Moreover, show that cusps will appear and disappear periodically along the string.

d) Explain why cusps form generically in 3+1 dimensions but not so in higher dimensions.

3.1. Light cone tensors

We want to derive some relations between Lorentz tensors $L_{\mu\nu}$ and light cone tensors.

a) Find the transformation matrix n^a_μ that transforms a Lorentz vector into light cone coordinates $X^a = n^a_\mu X^\mu$ where the index a runs through $+, -, 1, \ldots, D-2$ and μ is the usual spacetime index. Furthermore calculate the metric η_{ab} in light cone coordinates and show that

$$X_+ = -\frac{1}{2}X^-, \qquad X_- = -\frac{1}{2}X^+.$$

b) Show that the equality of components of the Lorentz tensors $A_{\mu_1\mu_2\ldots\mu_n} = B_{\mu_1\mu_2\ldots\mu_n}$ implies the equality of the light cone tensor components, i.e.

$$A_{++\ldots+} = B_{++\ldots+}, \quad A_{++\ldots-} = B_{++\ldots-}, \quad \ldots \quad A_{--\ldots-} = B_{--\ldots-},$$

and give explicitly the light cone components L_{++}, L_{+-}, L_{-+}, L_{--} in terms of the components of a Lorentz 2-tensor $L_{\mu\nu}$.

c) Furthermore, show that the trace of a rank-2 light cone tensor A is given by

$$A^\mu{}_\mu = -2A_{+-} - 2A_{-+} + A_{ii}.$$

3.2. Light cone gauge and mode expansion

Using our newly found knowledge about light cone tensors, we will investigate the form of the angular momentum generator. The mode expansion is given by

$$X^\mu(\sigma,\tau) = x_0^\mu + \kappa^2 p^\mu \tau + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-in(\tau+\sigma)}.$$

a) In the earlier problem 2.1a) you derived the angular momentum current $\mathcal{J}^{\mu\nu}_\alpha$ and its conserved charge

$$J^{\mu\nu} = \int_0^{2\pi} d\sigma\ \mathcal{J}_0^{\mu\nu}.$$

Express $\mathcal{P}_0^\mu$ in terms of the mode expansion. Calculate $J^{\mu\nu}$ in terms of the mode expansion. *Hint:* Contemplate the meaning of *conserved* quantity and use

$$\int_0^{2\pi} d\sigma\, e^{in\sigma} = 2\pi\delta_{n,0}.$$

b) Express J^{-i} in terms of the above derived mode expansion for the Lorentz tensor $J^{\mu\nu}$. In a quantum field theory, symmetry generators should be realised by hermitian operators

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}.$$

Assume canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, and show that J^{-i} is *not* hermitian. "Hermiticise" the generator.

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3.3. Maxwell and Kalb–Ramond fields

Light cone gauge is not only useful in string theory to extract physical information. It also is a valid gauge in other theories. First we will work on the Maxwell gauge field A_μ. Then we will turn to the Kalb–Ramond field $B_{\mu\nu}$ which will enter our description of quantum string theory in due course. (If you feel confident enough you can skip parts a) and b). Otherwise work carefully through all the subproblems for maximal benefit! You may find chapter 10 in Zwiebach – "A first course in String theory" useful.)

a) The Maxwell field $A_\mu(x)$ has a gauge symmetry

$$A'_\mu = A_\mu + \partial_\mu \epsilon(x).$$

We define the antisymmetric field strength tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that $F_{\mu\nu}$ is gauge invariant and derive the equations of motion for A_μ from the action (leaving coupling constants aside)

$$S_{\mathrm{YM}} = -\frac{1}{4}\int d^D x\, F_{\mu\nu}F^{\mu\nu}.$$

Rewrite the equations of motion in momentum space $\partial_\mu \leftrightarrow p_\mu$.

b) We want to implement light cone gauge. Express the gauge transformation in momentum space. Show that, by a sensible choice of $\epsilon(p)$, you can gauge away the $+$-component of the light cone gauge field (A_+, A_-, A_i) and deduce that the equation of motion in momentum space drastically simplifies in this gauge. Count the total number of *independent* degrees of freedom of the gauged Maxwell field.

The Kalb–Ramond field $B_{\mu\nu}$ is an antisymmetric Lorentz tensor with the gauge symmetry transformation

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu.$$

We define a field strength and an action for $B_{\mu\nu}$ by

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \quad \text{and} \quad S_{\mathrm{KR}} = -\frac{1}{12}\int d^D x\, H_{\mu\nu\rho}H^{\mu\nu\rho}.$$

c) Show that the gauge transformation of $B_{\mu\nu}$ has a redundancy

$$\epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda$$

under which $\delta B_{\mu\nu}$ is invariant. Express the gauge transformations in light cone momentum space and show that you can gauge away the component ϵ_+, such that the effective gauge transformation of $B_{\mu\nu}$ is generated by ϵ_- and ϵ_i.

d) Go through the steps in a), b) for $B_{\mu\nu}$ and $H_{\mu\nu\rho}$ – bearing in mind the result of c) – and show that the Kalb–Ramond field has only one independent degree of freedom in four dimensions.

e) In four dimensions, we can define a "dual field" $\bar{H}_\mu$ by contracting the field strength $H^{\mu\nu\rho}$ with the totally antisymmetric tensor of fourth order

$$\bar{H}_\mu = \varepsilon_{\mu\nu\rho\kappa} H^{\nu\rho\kappa}.$$

Using the result you found in the last part of this problem show that the dual field can be expressed by the derivative of a single scalar field. What does this imply for the Kalb–Ramond field in four dimensions?

4.1. Virasoro algebra

In this exercise we want to investigate in detail the Virasoro algebra as it appears in light cone string theory. For simplicity we will only work with the left movers L_n^{L} as the right movers L_n^{R} commute with these and satisfy an identical algebra. The mode operators α_n^i (with $i = 1, \ldots, D-2$) satisfy the algebra (we drop the L/R superscript)

$$[\alpha_n^i, \alpha_m^j] = m\delta^{ij}\delta_{n+m},$$

and the *normal ordered* Virasoro generators are given by

$$L_n = \frac{1}{2}\sum_{p\geq 0} \alpha_{n-p}^i \alpha_p^i + \frac{1}{2}\sum_{p<0} \alpha_p^i \alpha_{n-p}^i.$$

An algebra $\mathfrak{g}$ is a Lie algebra if its product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) is antisymmetric $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$ and satisfies the Jacobi identity

$$\big[a, [b, c]\big] + \big[b, [c, a]\big] + \big[c, [a, b]\big] = 0 \quad \text{for all } a, b, c \in \mathfrak{g}.$$

a) Show that the commutator of two Virasoro generators with $m + n \neq 0$ is given by

$$[L_m, L_n] = \frac{1}{2}\sum_p p\, \alpha_{m-p}^i \alpha_{p+n}^i + (m-p)\alpha_{n+m-p}^i \alpha_p^i.$$

b) By relabelling the summands, rewrite the above result in the following form

$$[L_m, L_n] = (m-n)L_{n+m}.$$

Argue that the complete solution, including the terms $n = -m$ is given by

$$[L_m, L_n] = (m-n)L_{m+n} + C(m)\delta_{m+n,0},$$

where $C(m)$ is a real valued, odd ($C(-m) = -C(m)$) function. The last term is called the *central extension* of the Virasoro algebra. Determine $C(m)$ up to two constants by considering the Jacobi identity. The solution is given by $C(m) = \frac{1}{12}(D-2)(m^3 - m)$.

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4.2. Analytical continuation of the zeta-function

In the lecture you have seen the peculiar result of the sum $\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$. In this exercise we will try to understand where the result comes from by analytically continuing the ζ-function. The Γ and ζ functions of a complex variable z are given by

$$\Gamma(z) = \int_0^{\infty} dt\, e^{-t} t^{z-1} \quad \text{and} \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

a) We start by regularising $\zeta(-1)$ using a small parameter ϵ. Show that we can write the zeta function like $\zeta_\epsilon(-1) = -\frac{\partial}{\partial\epsilon}\sum_{n=1}^{\infty} e^{-n\epsilon}$ in the limit $\epsilon \to 0$. Argue that the sum in this expression is convergent and give the solution. Expand the expression for small ϵ and show that the result is given by $\zeta_\epsilon(-1) \approx \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)$.

b) Show that for $\mathrm{Re}(z) > 1$ you can write

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{dt\, t^{z-1}}{e^t - 1}.$$

Conclude that it is possible to rewrite the integral to give

$$\Gamma(z)\zeta(z) = \int_0^1 dt\, t^{z-1}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{z-1} - \frac{1}{2z} + \frac{1}{12(z+1)} + \int_1^{\infty} \frac{dt\, t^{z-1}}{e^t - 1}.$$

c) The right hand side is well defined for $\mathrm{Re}(z) > -2$ (why?). We know that $\Gamma(z)$ has poles for $z = 0, -1, -2, \ldots$ with residues

$$\mathrm{Res}_{z_0=-n}\big[\Gamma(z_0)\big] = \frac{(-1)^n}{n!}.$$

Conclude that the values of $\zeta(z)$ at $z = 0$ and $z = -1$ are

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12}.$$

4.3. Poincaré transformations

Poincaré transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$ form a group whose product is defined as $T(\Lambda_1, a_1)T(\Lambda_2, a_2) = T(\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$. The inverse reads $T(\Lambda, a)^{-1} = T(\Lambda^{-1}, -\Lambda^{-1}a)$. Consider an infinitesimal transformation with generators J and P

$$T(1+\omega, \epsilon) = 1 + \frac{i}{2}\omega^{\mu\nu} J_{\mu\nu} - i\epsilon^\mu P_\mu + \ldots.$$

They define the Lie algebra of the Poincaré group. Show that

$$T(\Lambda, a)\, T(1+\omega, \epsilon)\, T(\Lambda, a)^{-1} = T\big(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\big).$$

How do J and P transform under $T(\Lambda, a)$? What relations do you get when you take Λ, a to be infinitesimal as well?

5.1. Lorentz invariance in light cone gauge

Since Lorentz invariance is obscured due to the light cone gauge it is not obvious that the following commutator vanishes

$$[\mathcal{J}^{-i}, \mathcal{J}^{-k}] \overset{?}{=} 0\,.$$

This exercise sheet will be solely concerned with the calculation of this commutator and its physical implications for bosonic string theory. In a previous sheet the generator was determined to be given by (up to a doubling of the latter term due to left and right movers)

$$\mathcal{J}^{-i} = \frac{1}{2}(p^- x_0^i + x_0^i p^-) - x_0^- p^i + i\sum_{n=1}^{\infty}(\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^-).$$

Furthermore, we have

$$p^- = \frac{L_0}{2\kappa^2 p^+} \quad \text{and} \quad \alpha_n^- = \frac{\sqrt{2}}{\kappa p^+} L_n$$

with L_n the Virasoro generators from the previous sheets. We depart here from the definition of sheet 4 and account for the normal ordering ambiguity in L_0 by defining

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - a,$$

where a is the so called intercept. This means we have two free parameters to adjust during this calculation: the intercept a and the dimension D of spacetime. *There are many subtleties in this calculation. Careful checks after every step are recommended.*

a) Begin by calculating all the possible commutators between the zero modes p^+, p^-, p^i, x_0^-, x_0^i and the α_n^i, α_n^- modes using the given commutators

$$[p^+, x_0^-] = i, \quad [p^j, x^k] = -i\delta^{jk}, \quad [\alpha_m^i, \alpha_n^j] = m\delta_{m+n}\delta^{ij}.$$

Hint: Watch out for subtleties when it comes to commutators with p^- and α^-. The commutator $[\alpha_m^-, \alpha_n^-]$ is the hardest here. However, you know its naive form already from the last sheet. Remember that there is a normal ordering ambiguity for α_0^-!

b) Calculate the commutator. The expected result is

$$[\mathcal{J}^{-i}, \mathcal{J}^{-j}] = 2\left(p^- - \frac{\sqrt{2}}{\kappa}\alpha_0^-\right)\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}^{[i}\alpha^{j]}{}_n$$
$$+ \frac{2}{\kappa^2 (p^+)^2}\sum_{n=1}^{\infty}\left(\left[\frac{D-2}{12} - 2\right]n + \frac{1}{n}\left[2a - \frac{D-2}{12}\right]\right)\alpha_{-n}^{[i}\alpha_n^{j]}.$$

For general values of a and D we say that the symmetry is *anomalous* because the right hand side is not zero. However, we can make it vanish. What are the reasons for the first term and conditions for the second term in this expression to vanish?

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5.2. T-duality: self-dual radius

When compactifying one dimension of string theory on a circle, new stringy effects appear that cannot be seen in field theory. One of these effects is *winding* of the string around the compactified dimension. We compactify the coordinate $D-2$ on a circle (drop the index $D-2$ for simplicity) and request that

$$X^{D-2}(\sigma+2\pi,\tau) = X^{D-2}(\sigma,\tau) + 2\pi\kappa^2 w$$

where $w = mR/\kappa^2$ is the *winding*. The left and right movers are then defined as usual for all directions except for the compact dimension where

$$X_{\mathrm{L}} = \frac{1}{2}x + \frac{\kappa^2}{2}(p+w)\xi^{\mathrm{L}} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^{\mathrm{L}}}{n}\exp(-in\xi^{\mathrm{L}}),$$
$$X_{\mathrm{R}} = \frac{1}{2}x + \frac{\kappa^2}{2}(p-w)\xi^{\mathrm{R}} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^{\mathrm{R}}}{n}\exp(-in\xi^{\mathrm{R}}).$$

a) Derive the $(D-1)$-dimensional mass-squared $M^2 = -p^2$ of states in the presence of a compact dimension in terms of the level operators N^{L} and N^{R}, the winding number m and compact momentum $p = n/R$. *Hint:* Use the Virasoro constraints $L_0^{\mathrm{L}} = L_0^{\mathrm{R}} = a$.

b) Show that the level matching constraint $N^{\mathrm{L}} = N^{\mathrm{R}}$ does not hold for strings with both winding number m and Kaluza–Klein momentum number n not equal zero. What happens to these states at $R \to \infty$?

c) Consider the mass formula for the cases

$$m = n = 0; \quad m = 0, n \neq 0; \quad m \neq 0, n = 0; \quad m = n = \pm 1; \quad m = -n = \pm 1.$$

For which values of N^{L} and N^{R} does the spectrum (possibly) contain tachyonic and massless states? What is their spin (scalar, vector, tensor) as viewed from $D-1$ non-compact spacetime dimensions?

d) In the case of non-zero winding and non-zero compact momentum ($m = n = \pm 1$ and $m = -n = \pm 1$) show that there is a special radius R^* where some states become massless. What happens for the other cases at this radius?

Introduction to String Theory — Problem Set 6

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6.1. Stretched strings

An open string can stretch between two Dp-branes. In fact there are four possibilities for a string to stretch between two branes, called *sectors*. Two of the sectors are strings beginning and ending on the same branes denoted [11] and [22]. The sectors [12] and [21] contain the cases where the string stretched between the brane. The last two cases are different because orientation matters. We will be interested in the case where the endpoints of the string lie on two different branes.

a) Write down the mode expansion for a string stretched between two parallel Dp-branes and interpret the result.

b) How does the distance between the branes affect the spectrum of the string? What happens for coincident branes? *Hint:* Consider the mass-squared.

6.2. Orbifolds

After having seen how a string behaves under compactification of one dimension, we want to find out how it behaves under restricting it to the half line $x^{D-1} \geq 0$ by the identification

$$x^{D-1} \sim -x^{D-1}.$$

Such a space is called *orbifold* in string theory. Again we abbreviate the relevant coordinate $X(\xi) := X^{D-1}(\xi)$, and introduce an operator U acting as ($\mu = 0, \ldots D-2$)

$$UX(\sigma)U^{-1} = -X(\sigma+\pi), \qquad UX^{\mu}(\sigma)U^{-1} = X^{\mu}(\sigma+\pi).$$

U is a symmetry of the orbifold theory, so only states invariant under U are physical.

a) How does U act on the modes x, p, α_n^{L} and α_n^{R} of the coordinate X?

b) Define the string vacuum $|0; q^{\mu}, r\rangle$ where $\mu = 0, \ldots, D-2$ and r is the momentum in the folded dimension. We assume that

$$U|0; q^{\mu}, 0\rangle = |0; q^{\mu}, 0\rangle.$$

Give the action of U on $|0; q^{a}, r\rangle$ and write down the ground states of the orbifold theory.

c) What are the massless states of the orbifold theory?

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6.3. Two-point function

In this exercise we want to compute the closed string propagator

$$\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle = -\frac{\kappa^2}{2}\eta^{\mu\nu}\log|z-z'|^2$$

which is given by the difference of the time-ordered and the normal ordered product of the operators $X^\mu(z,\bar{z})$ and $X^\nu(z',\bar{z}')$. Assume $|z|>|z'|$. *Hint:* You may ignore the effects of the centre of mass coordinates x^μ or use $:p^\mu x^\nu: = x^\nu p^\mu$.

6.4. Conformal transformations

Consider conformal transformations $z \to z'(z)$. Primary fields transform as tensors under conformal transformations

$$\mathcal{O}'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}(z'(z),\bar{z}'(\bar{z}))$$

a) How does a primary field transform under infinitesimal transformations $z' \to z+\zeta(z)$?

b) Show that the operator $:e^{ikX}:$ for a single scalar field X is primary by computing its OPE with the stress-energy tensor. Determine the conformal weights h and $\bar{h}$.

7.1. The complex logarithm

The propagator that you calculated above and the two-point function

$$\big\langle \partial X^\mu(z,\bar z)\bar\partial X^\nu(w,\bar w)\big\rangle = \pi\kappa^2\eta^{\mu\nu}\delta^2(z-w,\bar z-\bar w)$$

are related by the two derivatives. Show that

$$\partial\bar\partial \log|z|^2 = 2\pi\delta^2(z,\bar z)$$

a) ... by considering the divergence theorem

$$\int_{D^1} dz d\bar z(\partial v^z + \bar\partial v^{\bar z}) = i\oint_{\partial D^1}(d\bar z v^z - dz v^{\bar z}),$$

b) ... by regulating the singularity at $z = 0$ by

$$\partial\bar\partial\log|z|^2 := \lim_{\epsilon\to 0}\partial\bar\partial\log(|z|^2+\epsilon^2).$$

where $\epsilon > 0$ is a small parameter.

7.2. Schwarzian derivative

The stress-energy tensor transforms under a finite conformal transformation $z \to z' = f(z)$ as

$$T(z) \to T'(z) = (\partial f)^2 T(z') + \frac{c}{12}\,S(z',z)$$

where

$$S(z',z) = \frac{\partial f(z)\partial^3 f(z) - \frac{3}{2}(\partial^2 f(z))^2}{(\partial f)^2}$$

is the Schwarzian derivative.

a) Show that the Schwarzian derivative reproduces the correct infinitesimal transformation.

b) Show that the Schwarzian derivative has the correct property under successive conformal transformations, i.e. $x \to x' \to x''$ and $x \to x''$ yield the same transformation of T.

c) Prove that for a transformation

$$f(z) = \frac{az+b}{cz+d}$$

the Schwarzian derivative yields $S(z',z) = 0$. Why is this not surprising?

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7.3. Virasoro algebra from stress-energy OPE

The commutation relations for the Virasoro algebra can be derived from the stress-energy tensor $T(z)$. After Laurent expanding, the stress-energy tensor reads

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}.$$

Convince yourself that the Virasoro generators L_m can be written as

$$L_m = \frac{1}{2\pi i} \oint dz\, z^{m+1} T(z).$$

Calculate the commutators $[L_m, L_n]$ making use of the stress-energy OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots .$$

Hint: Think about along which contours you have to integrate and the meaning of radial ordering. Once done so, expand z^{m+1} in terms of the second insertion point w.

8.1. Veneziano amplitude

The Veneziano amplitude led to the discovery of string theory. In this problem we will attempt to calculate this amplitude. The open string tachyon vertex operator is given by an integral over the boundary of the string

$$V(k) = \sqrt{g_s} \int dx \, {:}\exp(ik_\mu X^\mu(x)){:}$$

such that the four tachyon scattering amplitude $A_4(k_1, k_2, k_3, k_4)$ is given by the expression

$$A_4(k_1, k_2, k_3, k_4) \sim \frac{1}{g_\mathrm{s}} \big\langle V_1 \ldots V_4 \big\rangle \sim g_\mathrm{s} \int\limits_{x_i < x_{i+1}} \prod_{i=1}^{4} dx_i \big\langle {:}e^{ik_1 \cdot X(x_1)}{:} \ldots {:}e^{ik_4 \cdot X(x_4)}{:} \big\rangle.$$

The ordering of insertions x_i is due to Chan–Paton factors.

a) The expectation value is computed using Wick's theorem and the two-point correlator

$$\big\langle X^\mu(x) X^\nu(y) \big\rangle = -2\kappa^2 \eta^{\mu\nu} \log|x - y|.$$

As a first step to calculate the expectation value apply Wick's theorem to only two insertions, i.e. show that

$${:}e^{ik_i \cdot X(x_i)}{:}\,{:}e^{ik_j \cdot X(x_j)}{:} = e^{\langle (ik_i \cdot X(x_i))(ik_j \cdot X(x_j)) \rangle} {:}e^{ik_i \cdot X(x_i) + ik_j \cdot X(x_j)}{:}.$$

b) Now calculate the expectation value for four points

$$\big\langle {:}e^{ik_1 \cdot X(x_1)}{:}\,{:}e^{ik_2 \cdot X(x_2)}{:}\,{:}e^{ik_3 \cdot X(x_3)}{:}\,{:}e^{ik_4 \cdot X(x_4)}{:} \big\rangle \, .$$

Hint: After doing that you should obtain the following integral expression for A_4 (up to the momentum-conserving delta function which is more subtle)

$$A_4 \sim g_s \delta^{26}(k_1 + k_2 + k_3 + k_4) \int \prod_{i=1}^{4} dx_i \prod_{j<l} |x_j - x_l|^{2\kappa^2 k_j \cdot k_l}.$$

c) Show that the integral is invariant under the $\mathrm{SL}(2, \mathbb{R})$ Möbius transformation

$$x_i \to \frac{ax_i + b}{cx_i + d}$$

for on-shell momenta $k_i^2 = \kappa^{-2}$. *Hint:* Use momentum conservation $\sum_i k_i = 0$.

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d) The integral given above is divergent because it has the non-compact Möbius group as a symmetry. It thus contains an irrelevant factor of the group volume which is infinite. We divide by the latter and use the symmetry to set $x_1 = 0$, $x_2 = x$, $x_3 = 1$ and $x_4 \to \infty$. Explain why the amplitude after the transformation reduces to

$$A_4 \sim g_s \delta^{26}(k_1 + k_2 + k_3 + k_4) \int dx \; |x|^{2\kappa^2 k_1 \cdot k_2} |1 - x|^{2\kappa^2 k_2 \cdot k_3} + (k_2 \leftrightarrow k_3).$$

What is the integration range of x_2 now? Why? What happened to the normalisation in front of the integral?

e) The resulting integral is well known. It is in the form of the Euler beta function

$$\mathrm{B}(a,b) = \int_0^1 dy \; y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,.$$

Write down the solution for the amplitude. *Hint:* Use the Mandelstam variables

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2$$

to simplify the result.

f) Where does this amplitude have poles? What do these poles correspond to?

9.1. Gravity background and renormalisation

So far we only considered flat Minkowski space. From GR we know however that gravity causes curved spacetime. Consider now the Polyakov action in conformal gauge where the flat metric tensor is replaced by a general coordinate-dependent one, $\eta_{\mu\nu} \to G_{\mu\nu}(X)$,

$$S = \frac{1}{4\pi\kappa^2} \int d^2\xi \, G_{\mu\nu}(X) \partial_\alpha X^\mu \partial^\alpha X^\nu .$$

a) Consider small fluctuations about the constant solution $X^\mu(\xi) = X_0^\mu + \kappa Y^\mu(\xi)$. Expand the action up to fourth order in Y. *Hint:* Use that locally you can always choose Riemann normal coordinates

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{\kappa^2}{3} R_{\mu\lambda\nu\kappa}(X_0) Y^\lambda Y^\kappa + \mathcal{O}(Y^3).$$

We can now study this as an interacting QFT of the fields Y. Let us look at the correction to the two-point correlator

$$\langle 0|Y^\lambda(\xi_1) Y^\kappa(\xi_2)|0\rangle_{\text{int}} = \frac{\langle 0|Y^\lambda(\xi_1) Y^\kappa(\xi_2) \, \exp S_{\text{int}}[Y]|0\rangle}{\langle 0|\exp S_{\text{int}}[Y]|0\rangle}$$

with

$$S_{\text{int}}[Y] = -\frac{1}{4\pi} \frac{\kappa^2}{3} R_{\mu\omega\nu\rho}(X_0) \int d^2\xi \, Y^\omega(\xi) Y^\rho(\xi) \partial_\alpha Y^\mu(\xi) \partial^\alpha Y^\nu(\xi) + \mathcal{O}(Y^5) .$$

At first order in the exponent you get a divergent contribution from a tadpole diagram. This divergence is caused by the propagator at coincident points $\xi \to \xi'$

$$\langle 0|Y^\mu(\xi) Y^\nu(\xi')|0\rangle = -\frac{1}{2}\eta^{\mu\nu} \ln|\xi - \xi'|^2 = 2\pi\eta^{\mu\nu} \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik(\xi-\xi')}}{k^2} .$$

It can however be absorbed by renormalisation.

b) To give you a taste of renormalisation, let us first isolate the UV singularity of the integral in dimensional regularisation

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 + m^2} .$$

(The mass is only introduced so that you do not have to worry about IR divergences.)
Hint: This is the recipe:

- Go to spherical coordinates and use $\int d\Omega_D = 2\pi^{D/2} / \Gamma(D/2)$.
- Rewrite the radial part such that you recognise

$$\frac{\Gamma(a)\,\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} = \int_0^\infty dy \, y^{\alpha-1}(1+y)^{-\alpha-\gamma} .$$

- Write $D = 2 - \epsilon$ and expand in ϵ. Use

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} + \Gamma'(1) + \mathcal{O}(\epsilon).$$

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c) Now we can renormalise the theory by introducing a counterterm into the action that removes the divergent part

$$\frac{\kappa^2}{12\pi\epsilon}\, R_{\mu\nu}\, \partial_\alpha Y^\mu\, \partial^\alpha Y^\nu.$$

Show that you can absorb the counterterm in a renormalisation of the field and the metric in the leading-order action

$$Y^\mu \to Y^\mu - \frac{\kappa^2}{6\epsilon}\, R^{\mu\nu} Y_\nu, \qquad G_{\mu\nu} \to G_{\mu\nu} + \frac{\kappa^2}{\epsilon}\, R_{\mu\nu}.$$

d) What does this imply for the beta function $\beta_{\mu\nu}(G)$?

9.2. Low-energy effective action

In the string frame the low-energy effective action is given by

$$S = \frac{1}{2\kappa^2}\int d^{26}X \sqrt{-\det G}\, e^{-2\Phi}\Big(R - \tfrac{1}{12} H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\, \partial^\mu\Phi\Big).$$

Here $G_{\mu\nu}$ is the metric, R the associated Ricci scalar, $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$ is the Kalb–Ramond field strength and Φ is a scalar, the dilaton field.

a) Show that the equations of motion of these fields are equivalent to the vanishing of the β functions

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \kappa^2 R_{\mu\nu} + 2\kappa^2 D_\mu D_\nu \Phi - \tfrac{1}{4}\kappa^2 H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma},\\
\beta_{\mu\nu}(B) &= -\tfrac{1}{2}\kappa^2 D^\lambda H_{\lambda\mu\nu} + \kappa^2 D^\lambda\Phi\, H_{\lambda\mu\nu},\\
\beta(\Phi) &= -\tfrac{1}{2}\kappa^2 D^\mu D_\mu \Phi + \kappa^2 D_\mu\Phi D^\mu\Phi - \tfrac{1}{24}\kappa^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}.
\end{aligned}$$

b) The kinetic energy term of the dilaton in the action seems to have the wrong sign. Explain why this is not so.

10.1. Linear dilaton

The general worldsheet action for massless background fields is given by

$$S = \frac{1}{4\pi\kappa^2} \int d^2\xi \Big(\big(\sqrt{-\det g}\, g^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \big) \partial_\alpha X^\mu \partial_\beta X^\nu + \kappa^2 \sqrt{-\det g}\, R[g]\, \Phi \Big).$$

To consider a concrete example of a background for string theory we want to have a look at the *linear dilaton background* where

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad B_{\mu\nu} = 0 \qquad \text{and} \qquad \Phi = V_\mu X^\mu$$

with V_μ a constant vector and $R[g]$ the worldsheet Ricci scalar.

a) Show that the beta functions defined on the last problem sheet vanish for $V_\mu V^\mu = (26 - D)/6\kappa^2$.

b) Derive the holomorphic worldsheet stress-energy tensor

$$T(z) = -\frac{1}{\kappa^2} {:}\partial X^\mu \partial X_\mu{:} + V_\mu \partial^2 X^\mu$$

of this theory and show that the central charge is given by

$$c = D + 6\kappa^2 V_\mu V^\mu.$$

10.2. Worldsheet supersymmetry

An action with global worldsheet supersymmetry is given by

$$S = -\frac{1}{4\pi\kappa^2} \int d^2\xi (\partial^\alpha X^\mu\, \partial_\alpha X_\mu + \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu).$$

The Grassmann-valued fields Ψ^μ are two-dimensional Dirac spinors. The ρ^α are 2×2 matrices satisfying the Clifford algebra

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$$

and the Dirac conjugate is $\bar{\Psi} = i\Psi^\dagger \rho^0$ with representation

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a) Show that this action is invariant under $\mathcal{N} = 1$ supersymmetry

$$\delta X^\mu = \bar{\epsilon} \Psi^\mu, \quad \delta \Psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon.$$

b) Evaluate the commutators $[\delta_1, \delta_2] X^\mu$ and $[\delta_1, \delta_2] \Psi^\mu$ to show that the commutator of two supersymmetry transformations amount to a translation along the worldsheet.

c) Derive the Noether current (supercurrent) of supersymmetry transformations.

d) Explain why you can decompose the Dirac spinors into real chiral (Majorana–Weyl) spinors. Write the Lagrangian in terms of these components.

e) Find the equations of motion for the chiral components. What are the boundary conditions that have to be satisfied in order to make the boundary term vanish?

You will find two distinct conditions. How does the mode expansion of the fermion fields look like in each case?

11.1. Kawai–Lewellen–Tye relations

In 1985, H. Kawai, D. C. Lewellen and S.-H. H. Tye discovered that closed string amplitudes can be expressed in terms of amplitudes in corresponding open string theories. Although the Kawai–Lewellen–Tye (KLT) relations connect the full string theory amplitudes, they are nowadays mostly used to relate low-energy limits of string theories. A prominent example for their application is the calculation of tree — and with some additional techniques as well loop — amplitudes in (supersymmetric) gravity theories from known amplitudes in (supersymmetric) gauge theories. We will use the conventional Regge slope parameter $\alpha' = \kappa^2$ to keep track of the orders in the low-energy expansion more easily.

a) The four-point tachyon amplitudes in open and closed string theory read

$$A^{\text{open}}_{4,\text{tach}}(s,t) \sim \frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)}{\Gamma(+2+\alpha' u)}$$

and

$$M^{\text{closed}}_{4,\text{tach}}(s,t) \sim \frac{\Gamma(-1-\alpha' s/4)\,\Gamma(-1-\alpha' t/4)\,\Gamma(-1-\alpha' u/4)}{\Gamma(+2+\alpha' s/4)\,\Gamma(+2+\alpha' t/4)\,\Gamma(+2+\alpha' u/4)}\,,$$

respectively, where the Mandelstam invariants s, t, u are defined in terms of the external momenta q_i as

$$s = -(q_1+q_2)^2, \quad t = -(q_1+q_4)^2, \quad u = -(q_1+q_3)^2.$$

The open and closed string tachyons come with mass $m^2 = -q_i^2 = -1/\alpha'$ and $m^2 = -q_i^2 = -4/\alpha'$ respectively.

Calculate the total mass square $s+t+u$ of the participating particles for the open and closed string amplitude. Using these results, convince yourself that the two amplitudes do indeed have the form of Euler Beta-functions. That is,

$$A^{\text{open}}_{4,\text{tach}} \sim \mathrm{B}(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)} \qquad \text{with} \qquad a+b=c,$$

$$M^{\text{closed}}_{4,\text{tach}} \sim \mathrm{B}(d,e,f) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(f)}{\Gamma(d+e)\,\Gamma(d+f)\,\Gamma(e+f)} \quad \text{with} \quad d+e+f=1.$$

b) Show explicitly, that the KLT-relation

$$M^{\text{closed}}_{4,\text{tach}}(s,t) \sim \frac{1}{\pi}\sin(\alpha'\pi t/4) A^{\text{open}}_{4,\text{tach}}(s/4,t/4) A^{\text{open}}_{4,\text{tach}}(t/4,u/4)$$

does indeed relate the amplitudes $A^{\text{open}}_{4,\text{tach}}$ and $M^{\text{closed}}_{4,\text{tach}}$. *Note:* The following relation is useful:

$$\Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin \pi a}\,.$$

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c) In order to relate massless open-string amplitudes $A_4^{\text{open}}(1,2,3,4)$ ("gluon amplitudes") to massless amplitudes $M^{\text{closed}}(1,2,3,4)$ in closed string theory ("graviton amplitudes"), one can write the KLT relations in the following form:

$$M_4^{\text{closed}}(1,2,3,4) = -\frac{i}{\alpha'\pi}\sin(\alpha'\pi s)\, A_4^{\text{open}}(1,2,3,4)\, A_4^{\text{open}}(1,2,4,3)\,,$$

where now, dealing with massless particles, $s+t+u=0$. Using the expansion of the four-point open-string amplitude

$$A_4^{\text{open}}(1,2,3,4) = A_4^{\text{YM}}(1,2,3,4)\Big(1-\alpha'^2\,\tfrac{1}{6}\pi^2 st + \alpha'^3\,\zeta_3\, stu + \mathcal{O}(\alpha'^4)\Big),$$

where A_4^{YM} denotes the low-energy limit $\alpha'\to 0$, show that the $\mathcal{O}(\alpha'^2)$-correction of the closed-string amplitude vanishes. Derive the $\mathcal{O}(\alpha'^3)$-correction and write the α'-expansion of the closed string four-point amplitude up to this order. Convince yourself of the total symmetry under exchange of the legs, as required for the gravity amplitude.

11.2. Number of independent open string amplitudes

Open string n-point gluon amplitudes are calculated from inserting appropriate vertex operators at a worldsheet with disk topology. While a naïve counting would imply $n!$ different amplitudes, there are two immediate symmetries built into the procedure: cyclicity and (worldsheet-) reflection

$$A_n^{\text{open}}(1,2,\ldots,n) = A_n^{\text{open}}(n,1,2,\ldots,n-1) \quad \text{and}$$
$$A_n^{\text{open}}(1,2,\ldots,n) = (-1)^n A_n^{\text{open}}(n,n-1,\ldots,2,1).$$

a) Convince yourself that the above relations reduce the number of independent open-string gluon amplitudes to $(n-1)!/2$.

b) In addition, there are further less obvious relations: the so-called monodromy relations

$$\begin{aligned}&A_n^{\text{open}}(1,2,3,4,\ldots,n) + e^{i\alpha'\pi s_{12}}A_n^{\text{open}}(2,1,3,4,\ldots,n)\\ &\quad + e^{i\alpha'\pi(s_{12}+s_{13})}A_n^{\text{open}}(2,3,1,4,\ldots,n)\\ &\quad + \ldots + e^{i\alpha'\pi(s_{12}+s_{13}+\ldots+s_{1,n-1})}A_n^{\text{open}}(2,3,4,\ldots,n-1,1,n) = 0\,,\end{aligned}$$

where $s_{ij} = -(q_i+q_j)^2$. Consider $\mathcal{O}(\alpha'^0)$ and $\mathcal{O}(\alpha'^1)$ in the α'-expansion of the monodromy relations for the four- and five-point amplitude. Show for those examples that the number of independent amplitudes reduces to $(n-3)!$. This is true for all n.

12.1. T-duality for the non-linear sigma-model: Buscher-rules

Consider the non-linear bosonic sigma-model in conformal gauge

$$S = -\frac{1}{4\pi\kappa^2}\int d^2\xi \left(\eta^{\alpha\beta}\, G_{\mu\nu} - \varepsilon^{\alpha\beta}\, B_{\mu\nu}\right) \partial_\alpha X^\mu\, \partial_\beta X^\nu\,,$$

where $\alpha, \beta = 1, 2$ are worldsheet indices and $\mu, \nu = 0, \ldots, 9$ are target space indices. We assume that the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$ are independent of the coordinate $X := X^9$, and we split the target space indices into the index 9 denoting the direction of the isometry as well as Latin indices $i, j = 0, \ldots, 8$ for the remaining dimensions.

The goal of this exercise is to calculate how T-duality along the direction X relates the original fields G and B with their T-dual counterparts $\tilde{G}$ and $\tilde{B}$. The original calculation was performed by T. H. Buscher in 1987.

This exercise is designed to guide you through a calculation which will lead to the action of the T-dual non-linear sigma-model. Written in terms of T-dual fields $\tilde{G}$ and $\tilde{X}$, the action can be brought into the same form as above, if the following Buscher rules are employed to relate original and T-dual fields:

$$\tilde{G}_{ij} = G_{ij} - \frac{G_{i9}G_{j9} - B_{i9}B_{j9}}{G_{99}}\,, \qquad \tilde{G}_{9i} = -\frac{B_{9i}}{G_{99}}\,, \qquad \tilde{G}_{99} = \frac{1}{G_{99}}\,,$$

$$\tilde{B}_{ij} = B_{ij} + \frac{G_{i9}B_{j9} - G_{j9}B_{i9}}{G_{99}}\,, \qquad \tilde{B}_{9i} = -\frac{G_{9i}}{G_{99}}\,.$$

a) Consider the derivatives of the coordinate X to be an independent field $V_\alpha = \partial_\alpha X$ and rewrite the above action in terms of the fields V_α and X^i. In order to ensure equality to the original action, a term containing the Lagrange multiplier $\tilde{X}$ needs to be added to the action:

$$-\frac{1}{2\pi\kappa^2}\int d^2\xi\, \varepsilon^{\alpha\beta}\tilde{X}\,\partial_\alpha V_\beta.$$

Convince yourself that the above action can be recovered from your action accompanied by the above term after making use of the equation of motion for $\tilde{X}$.

b) In order to find the T-dual form of the action, derive the equation of motion for the field V_α. You should find that

$$V_\alpha = \frac{1}{G_{99}}\,\eta_{\alpha\gamma}\Big[-\varepsilon^{\gamma\beta}\partial_\beta\tilde{X} - \big(\eta^{\gamma\beta}G_{9i} - \varepsilon^{\gamma\beta}B_{9i}\big)\partial_\beta X^i\Big]\,.$$

c) Using its equation of motion, eliminate the field V_α and rewrite the action including the term with the Lagrange multiplier in terms of dual coordinates $(X^i, \tilde{X})$. By doing so, you will have to rename several dummy indices. *Note:* The following identity is helpful:

$$\varepsilon^{\alpha\gamma}\eta_{\gamma\delta}\varepsilon^{\delta\beta} = \eta^{\alpha\beta}\,.$$

After collecting the terms into contributions proportional to $\partial_\alpha\tilde{X}\partial_\beta\tilde{X}$, $\partial_\alpha\tilde{X}\partial_\beta X^i$ and $\partial_\alpha X^i\partial_\beta X^j$, you should find that the T-dual fields $\tilde{G}$ and $\tilde{B}$ are indeed related to the original fields G and B by the Buscher rules.

13.1. S-duality in four-dimensional heterotic string theory

In this exercise, the $\mathrm{SL}(2,\mathbb{R})$-symmetry of the equations of motion in heterotic string theory compactified to four dimensions shall be considered. The reduction of the vector fields corresponding to the gauge groups $\mathrm{SO}(32)$ or $\mathrm{E}_8 \times \mathrm{E}_8$, however, leads to numerous vector fields and scalars in four dimensions. As it is not particularly enlightening to deal with the symmetry transformations of all those fields, let us instead consider a simpler example of an effective action governing the dynamics of the metric G_{S}, the antisymmetric tensor B, the dilaton Φ and just a single $\mathrm{U}(1)$ gauge field A_μ

$$S \sim \int d^4x \sqrt{-\det G_{\mathrm{S}}}\, e^{-2\Phi}\Big(R_{\mathrm{S}} + 4\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big),$$

where the subscript S refers to the metric in the string frame. Furthermore

$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} + \text{cyclic}) - \tfrac{1}{2}(A_\mu F_{\nu\rho} + \text{cyclic})$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

are the field strengths associated to the fields $B_{\mu\nu}$ and A_μ respectively. In the Einstein frame

$$G_{\mu\nu} = e^{-2\Phi} G_{\mathrm{S},\mu\nu}\,.$$

the equations of motion for the fields $G_{\mu\nu}$, $B_{\mu\nu}$, A_μ and the dilaton Φ read

$$\begin{aligned}
0 &= R_{\mu\nu} - 2D_\mu\Phi D_\nu\Phi - G_{\mu\nu}D^\rho D_\rho\Phi - \tfrac{1}{4}e^{-4\Phi}H_{\mu\rho\tau}H_\nu{}^{\rho\tau} - \tfrac{1}{2}e^{-2\Phi}F_{\mu\rho}F_\nu{}^\rho,\\
0 &= D_\rho\big(e^{-4\Phi}H^{\mu\nu\rho}\big),\\
0 &= D_\mu\big(e^{-2\Phi}F^{\mu\nu}\big) + \tfrac{1}{2}e^{-4\Phi}H_{\rho\mu}{}^\nu F^{\rho\mu},\\
0 &= D^\mu D_\mu\Phi + \tfrac{1}{12}e^{-4\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} - \tfrac{1}{8}e^{-2\Phi}F_{\mu\nu}F^{\mu\nu}\,.
\end{aligned}$$

a) The Bianchi identities arise as a consistency conditions for any field strength tensor. For our situation, the Bianchi identity for the field strength $H_{\mu\nu\rho}$ is given by

$$\big(\sqrt{-\det G}\big)^{-1}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = -\tfrac{3}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\,,$$

where the dual field strength $\tilde{F}_{\mu\nu}$ is defined as

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\big(\sqrt{-\det G}\big)^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,.$$

As has been shown already in a previous exercise, the field strength $H_{\mu\nu\rho}$ can be written in terms of a scalar field Ψ. Convince yourself that

$$H^{\mu\nu\rho} = -\big(\sqrt{-\det G}\big)^{-1}\varepsilon^{\mu\nu\rho\sigma}e^{4\Phi}\partial_\sigma\Psi$$

solves the equation of motion for $H_{\mu\nu\rho}$. Plug the above identity into the Bianchi identity to obtain a relation between the scalar Ψ and the field strength $F_{\mu\nu}$ and its dual.

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b) Define the complex field

$$\lambda = \Psi + ie^{-2\Phi} = \lambda_1 + i\lambda_2$$

and use the above identities as well as

$$F_\pm = F \pm i\tilde{F}$$

in order to rewrite the equations of motion and the result from the last subproblem in terms of λ and the fields $F_\pm$ only.

c) Convince yourself that the equations of motion written in the last subproblem are invariant under shifting λ by a real number

$$\lambda \to \lambda + r \quad \text{where} \quad r \in \mathbb{R}\,.$$

Show furthermore that the following set of transformations leaves the set of equations of motion invariant

$$\lambda \to -\frac{1}{\lambda}\,, \quad F_+ \to -\lambda F_+\,, \quad F_- \to -\bar{\lambda} F_-\,.$$

While invariance under the real shift is manifest for all equations of motion, under the second set of transformations two equations get interchanged. The equation of motion containing the Riemann tensor transforms into itself plus an extra term. In order for SL$(2,\mathbb{R})$ to be symmetry, this extra term should better vanish. While one can show that it indeed vanishes for a particular solution of the equation of motion, convince yourself that any SL$(2,\mathbb{R})$-transformation is just a scaling.

The transformations above generate the group SL$(2,\mathbb{R})$. Its action can be written as

$$\lambda \to \frac{a\lambda + b}{c\lambda + d} \quad \text{with} \quad ad - bc = 1 \quad \text{and} \quad F_+ \to -(c\lambda + d)F_+\,.$$

Thus the considered model exhibits an S-duality relating the weak- and strong-coupling regimes of the theory.