Quantum Information Theory
Solutions 12

HS 13
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Exercise 12.1 Entanglement and Teleportation

Quantum teleportation from Alice A to Bob B can be described by a linear map $\mathcal{E}$ from operators on $\mathcal{H}_A$ to operators on $\mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B$ are copies. In the lecture we showed that $\mathcal{E}[|\psi\rangle\langle\psi|] = |\psi\rangle\langle\psi|$ for all pure states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

1. *[a)] Look at the state of the system after Alice's measurement but before she communicates her results to Bob. At this point, we know that Bob's state is in one of four possible states that are related to $|\psi\rangle$. Show that we still cannot extract any information on $|\psi\rangle$ out of Bobs state by calculating it as a probabilistic mixture of the four possible states. What is the physical relevance of this observation?*

   Bob has one of the for states $|\phi_{ij}\rangle$, with $i, j \in \{0, 1\}$:

   $$|\phi_{ij}\rangle = \alpha|i\rangle + (-1)^j \beta|1-i\rangle$$

   or as density matrices:

   $$\rho_{ij} = |\alpha|^2|i\rangle\langle i| + |\beta|^2|i-1\rangle\langle i-1| + (-1)^j\alpha\bar{\beta}|i\rangle\langle 1-i| + (-1)^j\bar{\alpha}\beta|i-1\rangle\langle i|.$$

   Since Bob does not now the measurement outcome (i, j) he has to assume that all these states occur with equal probability. The density matrix describing his state is therefore given by

   $$\rho = \frac{1}{4}(\rho_{00} + \rho_{01} + \rho_{10} + \rho_{11}) = \frac{1}{2}\mathbb{1},$$

   as can easily be verified. Note that this state is independent of α and β and therefore cannot reveal any information about $|\psi\rangle$.

b) *Show that the pure states span the space of Hermitian matrices.*

Every hermitian matrix can be decomposed into projectors onto eigenvectors:

$$H = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|,$$

where λ_i are eigenvalues to the eigenvectors $|\phi_i\rangle$. Hence we can write every hermitian matrix as a linear combination of pure states.

c) *Show that $\mathcal{E}[\rho] = \rho$, for any mixed state ρ. Furthermore, show that $(\mathcal{E} \otimes \mathbb{1}_R)[|\Psi\rangle\langle\Psi|] = |\Psi\rangle\langle\Psi|$, for any $|\Psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_R$. This implies that quantum teleportation preserves entanglement!*

The operation $\mathcal{E}$ is assumed to be linear, from which follows that it is also well-defined for linear combinations of pure states. This obviosly includes mixed states, which are convex combinations of pure states. Any state on $\mathcal{H}_A \otimes \mathcal{H}_R$ can be written as:

$$\rho_{AR} = \sum_{i,j} x_{ij}\, h_A^i \otimes h_R^j,$$

where h_A^i, h_R^j form a basis of the hermitian matrices in their respective subspace, i.e. h^i are hermitian but not necessarily positive. For example the 2-dimensional space of hermitian matrices is spanned by the identity and the Pauli matrices. Obviously the tensor products of all pairs of basis matrices then form a basis of the larger space. Knowing this, we find

$$(\mathcal{E} \otimes \mathbb{1}_R)[\rho_{AR}] = \sum_{i,j} x_{ij}\, \mathcal{E}[h^i_A] \otimes h^j_R = \rho_{AR}.$$

This obviously also holds in the special case when ρ_{AR} is pure.

Exercise 12.2 Resource inequalities: teleportation and classical communication

We saw a protocol, teleportation, to transmit one qubit using two bits of classical computation and one ebit, $[qq] + 2[c \to c] \geq [q \to q]$. Now suppose that Alice and Bob share unlimited entanglement: they can use up as many ebits as they want. Can Alice send n qubits to Bob using less than $2n$ bits of classical communication? In other words, we want to know if the following is possible:

$$m[c \to c] + \infty[qq] \geq n[q \to q] + \infty[qq], \qquad m < 2n$$

Prove that this is not the case. **Hint:** *use superdense coding.*
We concatenate teleportation and superdense coding (with unlimited entanglement),

$$\begin{aligned} m[c \to c] + \infty[qq] &\geq n[q \to q] + \infty[qq] \\ n'[c \to c] + \infty[qq] &\geq m'[q \to q] + \infty[qq] \end{aligned}$$

We can fix $n = n'$. Superdense coding allows us to transmit two bits of classical information using one qubit and one ebit, so we have

$$\begin{array}{rcl} m[c \to c] & + & \infty[qq] \geq \\ n[c \to c] & + & \infty[qq] \geq \\ 2n[c \to c] & + & \infty[qq] \end{array}$$

Let us focus on the extremes of this resource inequality,

$$m[c \to c] + \infty[qq] \geq 2n[c \to c] + \infty[qq]$$

Our result follows immediately if we assume that $z[c \to c] + \infty[qq] \geq y[c \to c] + \infty[qq]$ implies $y \leq x$ (i.e., entanglement itself does not help us send more classical bits). The proof of that is similar to the one for quantum channels

Exercise 12.3 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.
Recall that we say that a bipartite state ρ_{AB} is separable (not entangled) if

$$\rho = \sum_k p_k\ \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

a) *Let $\Lambda_A : \mathrm{End}(\mathcal{H}_A) \mapsto \mathrm{End}(\mathcal{H}_A)$ be a positive map. Show that $\Lambda_A \otimes \mathcal{I}_B$ maps separable states to positive operators.*

   This means that if we apply $\Lambda_A \otimes \mathcal{I}_B$ to a bipartite state ρ_{AB} and obtain a non-positive operator, we know that ρ_{AB} is entangled. In other words, this is a sufficient criterion for entanglement.

If $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a separable state, it can be written as convex combination of product states, $\rho = \sum_i p_A \sigma_A^i \otimes \sigma_B^i$, and

$$\Lambda_A \otimes \mathbb{1}_B \left(\sum_i p_A \sigma_A^i \otimes \sigma_B^i \right) = \sum_i p_A \Lambda_A(\sigma_A^i) \otimes \sigma_B^i.$$

All $\{\Lambda_A \left(\sigma_A^i\right)\}_i$ are positive operators. Since the set of positive operators is convex, we know that a convex combination of positive operators is still positive.

b) *Apply the partial transpose, $\mathcal{T}_A \otimes \mathcal{I}_B$, to the ϵ-noisy Bell state*

$$\rho_{AB}^\epsilon = (1-\epsilon)\ |\psi^-\rangle\langle\psi^-| + \epsilon\ \frac{\mathbb{1}_4}{4}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad \epsilon \in [0,1].$$

For what values of ϵ can we be sure that ρ^ϵ is entangled?

Note: As shown earlier, the transpose map is defined by $\mathcal{T}\left(\sum_{ij} a_{ij} |i\rangle\langle j|\right) = \sum_{ij} a_{ji}\ |i\rangle\langle j|$,

Remark: Indeed, it can be shown that the PPT criterion (positive partial transpose) is necessary and sufficient for systems of dimension 2×2 and 2×3.

If $[\mathcal{I} \otimes \mathcal{T}](\rho^\epsilon)$ is not positive, ρ^ϵ is entangled (note: we have not proved the converse, though it is true too). We have

$$\begin{aligned}
\rho^\epsilon &= \frac{1-\epsilon}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| - |00\rangle\langle 11| - |11\rangle\langle 00|) + \frac{\epsilon}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|) \\
&= \frac{2-\epsilon}{4}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2-\epsilon}{4}|1\rangle\langle 1| \otimes |1\rangle\langle 1| - \frac{2-\epsilon}{4}|0\rangle\langle 1| \otimes |0\rangle\langle 1| - \frac{2-\epsilon}{4}|1\rangle\langle 0| \otimes |1\rangle\langle 0| \\
&\quad + \frac{\epsilon}{4}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{\epsilon}{4}|1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
&= \frac{1}{4}\begin{pmatrix} 2-\epsilon & 0 & 0 & 2-2\epsilon \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 2-\epsilon & 0 & 0 & 2-\epsilon \end{pmatrix} \\
[\mathcal{I} \otimes \mathcal{T}](\rho^\epsilon) &= \frac{2-\epsilon}{4}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2-\epsilon}{4}|1\rangle\langle 1| \otimes |1\rangle\langle 1| - \frac{2-2\epsilon}{4}|0\rangle\langle 1| \otimes |1\rangle\langle 0| - \frac{2-2\epsilon}{4}|1\rangle\langle 0| \otimes |0\rangle\langle 1| \\
&\quad + \frac{\epsilon}{4}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{\epsilon}{4}|1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
&= \frac{1}{4}\begin{pmatrix} 2-\epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 2-2\epsilon & 0 \\ 0 & 2-2\epsilon & \epsilon & 0 \\ 0 & 0 & 0 & 2-\epsilon \end{pmatrix}.
\end{aligned}$$

This has eigenvalues $\frac{1}{4}\{2-\epsilon, 2-\epsilon, 2-\epsilon, \epsilon-2\}$. The last eigenvalue is negative for $\epsilon \leq \frac{2}{3}$, so if the state is less than $\frac{2}{3}$-noisy, it is entangled.