Quantum Information Theory
Solutions 9

HS 13
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Exercise 9.1 The Choi-Jamiolkowski Isomorphism

Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \;\mapsto\; (1-\alpha)\,\frac{\mathbf{1}_2}{2} + \alpha\,\big(\frac{\mathbf{1}_2}{2} + X\rho Z + Z\rho X\big), \qquad 0 \le \alpha \le 1. \tag{1}$$

a) *Use the Bloch representation to determine for what range of α these mappings are positive. What happens to the Bloch sphere?*

The two-dimensional state space $\mathcal{S}(\mathcal{H}_2)$ is isomorphic to the unit sphere on $\mathbb{R}^3$:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \qquad x^2+y^2+z^2 \le 1.$$

We apply the mapping to this state and get

$$\rho' = \frac{1}{2}\begin{pmatrix} 1+2\alpha x & 2\alpha z \\ 2\alpha z & 1-2\alpha x \end{pmatrix}.$$

The mapping is trace-preserving, hence it is positive if and only if the determinant of ρ' is positive for all allowed values of x, y and z.

$$\det \rho' = \frac{1}{4}(1-4\alpha^2x^2-4\alpha^2z^2) \ge \frac{1}{4} - \alpha^2.$$

Hence, the mapping is positive for $0 \le \alpha \le \frac{1}{2}$.

b) *Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:*

$$\sigma_\alpha = (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

For what range of α is the mapping a CPM?

First, not that in the computational basis, we can write the fully entangled states as

$$|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1 \end{pmatrix}.$$

The mapping $\mathcal{E}_\alpha$ consists of a mapping to the identity and multiplications by the Pauli matrices X and Z. The latter operations can be seen as multiplications of $|\Psi\rangle\langle\Psi|$ from left and right by the matrices

$$X\otimes\mathbf{1}_2 = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \quad Z\otimes\mathbf{1}_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}$$

respectively. Hence the state σ_α corresponding to $\mathcal{E}_\alpha$ is given by

$$\begin{aligned}\sigma_\alpha &= (\mathcal{E}_\alpha \otimes \mathcal{I})\big[|\Psi\rangle\langle\Psi|\big] \\ &= \mathbf{1}_4 + \alpha\Big((X\otimes\mathbf{1}_2)|\Psi\rangle\langle\Psi|(Z\otimes\mathbf{1}_2) + (Z\otimes\mathbf{1}_2)|\Psi\rangle\langle\Psi|(X\otimes\mathbf{1}_2)\Big) \\ &= \frac{1}{4}\begin{pmatrix} 1 & 2\alpha & 2\alpha & 0 \\ 2\alpha & 1 & 0 & -2\alpha \\ 2\alpha & 0 & 1 & -2\alpha \\ 0 & -2\alpha & -2\alpha & 1 \end{pmatrix}.\end{aligned}$$

The eigenvalues σ_α are given by

$$\lambda_\alpha^1 = \frac{1}{4} - \alpha, \quad \lambda_\alpha^2 = \lambda_\alpha^3 = \frac{1}{4}, \quad \lambda_\alpha^4 = \frac{1}{4} + \alpha.$$

Therefore, the mapping $\mathcal{E}_\alpha$ is a CPM for $0 \le \alpha \le \frac{1}{4}$.

c) *Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.*

The operator-sum representation can be found via the isometry $\mathcal{U}_\alpha$ that corresponds to the purification of σ_α. To purify σ_α, let us first list its eigenvectors:

$$|\nu^1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad |\nu^2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad |\nu^3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\nu^4\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

They are independent of α. Alternatively, using the Schmidt decomposition, we can write

$$|\nu^i\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |\theta_0^i\rangle + |1\rangle \otimes |\theta_1^i\rangle\big)$$

as can easily be verified as all eigenvectors are completely mixed on the first subsystem. The reduced vectors are given by

$$|\theta_0^1\rangle = \frac{1}{\sqrt{2}}\big(-|0\rangle + |1\rangle\big), \quad |\theta_1^1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \quad |\theta_0^2\rangle = -|1\rangle, \quad |\theta_1^2\rangle = |0\rangle$$

$$|\theta_0^3\rangle = |0\rangle, \quad |\theta_1^3\rangle = |1\rangle, \quad |\theta_0^4\rangle = \frac{1}{\sqrt{2}}\big(-|0\rangle - |1\rangle\big), \quad |\theta_1^4\rangle = \frac{1}{\sqrt{2}}\big(-|0\rangle + |1\rangle\big)$$

The purification $|\Theta_\alpha\rangle$ of σ_α can now be written as

$$\begin{aligned}|\Theta_\alpha\rangle &= \sum_{i=1}^{4} \sqrt{\lambda_\alpha^i}|\nu^i\rangle \otimes |i\rangle_R \\ &= \frac{1}{\sqrt{2}}\sum_{j=0}^{1} |j\rangle \otimes \Big(\sum_{i=1}^{4} \sqrt{\lambda_\alpha^i}|\theta_j^i\rangle \otimes |i\rangle_R\Big).\end{aligned}$$

We introduced a reference system $\mathcal{H}_R$ with basis $\{|i\rangle_R\}_i, i \in \{1,2,3,4\}$ to do the purification. This defines an isometry $\mathcal{U}_\alpha$ from our original two-dimensional Hilbert space to a 8-dimensional Hilbert space including $\mathcal{H}_R$ as follows:

$$\mathcal{U}_\alpha : \rho \mapsto U\,\rho\,U^\dagger, \quad U = \sum_{j=0}^{1}\left(\sum_{i=1}^{4} \sqrt{\lambda_\alpha^i}|\theta_j^i\rangle \otimes |i\rangle_R\right)\langle j|.$$

The original CPM $\mathcal{E}_\alpha$ can be recovered by tracing out the reference system $\mathcal{H}_R$:

$$\mathcal{E}_\alpha(\rho) = \mathsf{tr}_R\Big(\mathcal{U}_\alpha(\rho)\Big).$$

Thus, we finally get a set of operators

$$M_k = \frac{1}{\sqrt{\lambda_\alpha^k}}\big(\mathbf{1}_2 \otimes \langle k|_R\big)U = |\theta_0^k\rangle\langle 0| + |\theta_1^k\rangle\langle 1|.$$

Or, explicitly

$$M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

such that

$$\mathcal{E}_\alpha : \rho \;\mapsto\; \sum_{k=1}^{4} \lambda_\alpha^k\, M_k\, \rho\, (M_k)^\dagger.$$

Exercise 9.2 Uncertainty relations

In Section 6.2 in the script, we consider the measurements (cf. (6.24) and (6.25))

$$\tilde{\Gamma}_x = W^*(\tilde{P}_x \otimes id)W \quad \textit{and} \tag{3}$$
$$\Gamma_z = W^*(id \otimes P_z)W, \tag{4}$$

with W, $\tilde{P}_x$ and P_z as defined in the script. Show that they can be written as (6.26) and (6.27), i.e.,

$$\Gamma_z = \frac{1}{2}\left(id + (-1)^z \cos 2\theta \sigma_z\right) \tag{5}$$
$$\tilde{\Gamma}_x = \frac{1}{2}\left(id + (-1)^x \sin 2\theta \sigma_x\right). \tag{6}$$

Starting with Γ_z, we find

$$\Gamma_z = W^*(\mathbb{1} \otimes P_z)W \tag{7}$$
$$= \sum_{z'z''} |z'\rangle\langle z'|z''\rangle\langle z''|\langle\varphi_{z'}|z\rangle\langle z|\varphi_{z''}\rangle \tag{8}$$
$$= \sum_{z'} |z'\rangle\langle z'|\; |\langle\varphi_{z'}|z\rangle|^2 \tag{9}$$

Expressed as matrices in the $|z\rangle$ basis, the operators are

$$\Gamma_0 = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \text{and} \qquad \Gamma_1 = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \cos^2\theta \end{pmatrix}. \tag{10}$$

Clearly the two form a valid POVM. In terms of Pauli operators, the POVM elements are

$$\Gamma_z = \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \cos 2\theta\, \sigma_z\right) \tag{11}$$

which highlights the fact that Γ is sort of a noisy version of a measurement in the σ_z basis. Indeed, if $\theta = 0$, then Γ is just a measurement in that basis. The identity contributions to the POVM elements serve to make the two outcomes more equally-likely, i.e. they reduce the information the POVM collects about the σ_z basis.

For $\tilde{\Gamma}_x$ we have

$$\tilde{\Gamma}_x = (\sum_{z_1} |z_1\rangle\langle z_1| \otimes \langle\varphi_{z_1}|)(\tfrac{1}{2}\sum_{zz'}(-1)^{x(z+z')}|z\rangle\langle z'| \otimes \mathbb{1})(\sum_{z_2} |z_2\rangle\langle z_2| \otimes |\varphi_{z_2}\rangle) \tag{12}$$

$$= \tfrac{1}{2}\sum_{zz'}(-1)^{x(z+z')}|z\rangle\langle z'|\, \langle\varphi_z|\varphi_{z'}\rangle \tag{13}$$

Expressed as matrices in the $|z\rangle$ basis, we have

$$\tilde{\Gamma}_x = \tfrac{1}{2}\begin{pmatrix} 1 & (-1)^x \sin 2\theta \\ (-1)^x \sin 2\theta & 1 \end{pmatrix}. \tag{14}$$

Again this is a sort of noisy measurement, but now of the observable σ_x. In terms of Pauli operators, the $\tilde{\Gamma}_x$ take the form

$$\tilde{\Gamma}_x = \tfrac{1}{2}(\mathbb{1} + (-1)^x \sin 2\theta\, \sigma_x). \tag{15}$$

Exercise 9.3 "All-or-Nothing" Violation of Local Realism

Consider the three qubit state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)_{123}$, the Greenberger-Horne-Zeilinger state.

a) Show that $|\text{GHZ}\rangle$ is a simultaneous eigenstate of $X_1Y_2Y_3$, $Y_1X_2Y_3$, and $Y_1Y_2X_3$ with eigenvalue $+1$, where X and Y are the corresponding Pauli operators.

Observe that the three operators commute, since X and Y anticommute. Since the state is invariant under permutations of the three systems, we only need to check that it is an eigenstate of the first operator, since the others are generated from it by permutation. Both X and Y flip bits in the standard basis, but Y adds an extra $-i$ if the input is $|0\rangle$ and i if $|1\rangle$. Thus $XYY|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(-\text{i})^2|111\rangle - (\text{i}^2)|000\rangle = |\text{GHZ}\rangle$.

b) Use the results of part (a) to argue by Einstein locality that each qubit has well-defined values of X and Y. For qubit j, denote these values by x_j and y_j. We say that these values are *elements of reality*. What would local realism, i.e. the assumption of realistic values that are undisturbed by measurements on other qubits, predict for the product of the outcomes of measurements of X on each qubit?

Measuring Y on any two systems determines the X value on the third, so absent any "spooky action at a distance", the X value should be well-defined. Similarly, measurements of X and Y on any two determine the Y value of the third, so it should also be well-defined. For X measurements on each spin, the product $x_1x_2x_3 = 1$ since $x_1x_2x_3 = (x_1y_2y_3)(y_1x_2y_3)(y_1y_2x_3)$ (if x_j and y_k all take the values ± 1.)

c) What does quantum mechanics predict for the product of the outcomes of measurements of X on each qubit?

Measuring X on each system and taking the product is the same as measuring $X_1X_2X_3$. $|\text{GHZ}\rangle$ is clearly an eigenstate of this operator with eigenvalue -1, so $X_1X_2X_3 = -1$.