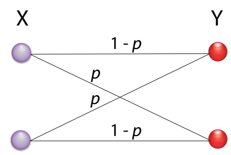
Quantum Information Theory
Solutions 8

Exercise 8.1 Classical channels as TPCPMs.

a) *Take the binary symmetric channel* $\mathbf{p}$,



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Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1 - q$, *we may express this as the 1-qubit mixed state* $\rho_X = q\ |0\rangle\langle 0| + (1-q)\ |1\rangle\langle 1|$.

What is the quantum state ρ_Y *that represents the final probability distribution* P_Y *in the computational basis?*

We have

$$P_Y(0) = \sum_x P_X(x) P_{Y|X=x}(0) = q(1-p) + (1-q)p$$
$$P_Y(1) = qp + (1-q)(1-p),$$

which can be expressed as a quantum state $\rho_y = [q(1-p)+(1-q)p]\ |0\rangle\langle 0| + [qp+(1-q)(1-p)]\ |1\rangle\langle 1| \in \mathcal{L}(\mathcal{H}_Y)$.

b) *Now we want to represent the channel as a map*

$$\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) \mapsto \mathcal{S}(\mathcal{H}_Y)$$
$$\rho_X \to \rho_Y.$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ *is a decomposition* $\{E_k\}_k$ *of operators* $E_k \in Hom(\mathcal{H}_X, \mathcal{H}_Y)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, *such that*

$$\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.$$

Find an operator-sum representation of $\mathcal{E}_{\mathbf{p}}$.

We take four operators, corresponding to the four different "branches" of the channel,

$$E_{0\to 0} = \sqrt{1-p}|0\rangle\langle 0|$$
$$E_{0\to 1} = \sqrt{p}|1\rangle\langle 0|$$
$$E_{1\to 0} = \sqrt{p}|0\rangle\langle 1|$$
$$E_{1\to 1} = \sqrt{1-p}|1\rangle\langle 1|.$$

To check that this works for the classical state ρ_X, we do

$$\begin{aligned}
\mathcal{E}(\rho_X) =& \sum_{xy} E_{x\to y}\ \rho_X\ E^\dagger_{x\to y} \\
=& \sum_{xy} E_{x\to y}\ \Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]\ E^\dagger_{x\to y} \\
=& (1-p)\ |0\rangle\langle 0|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|0\rangle\langle 0| \\
&+ p\ |1\rangle\langle 0|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|0\rangle\langle 1| \\
&+ p\ |0\rangle\langle 1|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|1\rangle\langle 0| \\
&+ (1-p)\ |1\rangle\langle 1|\Big[q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|\Big]|1\rangle\langle 1| \\
=& q(1-p)\ |0\rangle\langle 0| \\
&+ qp\ |1\rangle\langle 1| \\
&+ (1-q)p\ |0\rangle\langle 0| \\
&+ (1-q)(1-p)\ |1\rangle\langle 1| = \rho_Y.
\end{aligned}$$

c) *Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state ρ_X is not diagonal in the computational basis?*

In general, we can express the state in the computational basis as $\rho_X = \sum_{ij} \alpha_{ij}|i\rangle\langle j|$, with the usual conditions (positivity, normalization). Applying the map gives us

$$\begin{aligned}
\mathcal{E}(\rho_X) =& \sum_{xy} E_{x\to y}\ \Big[\sum_{ij} \alpha_{ij}|i\rangle\langle j|\Big]\ E^\dagger_{x\to y} \\
=& (1-p)\ |0\rangle\langle 0|\Big[\sum_{ij} \alpha_{ij}|i\rangle\langle j|\Big]|0\rangle\langle 0| \\
&+ p\ |1\rangle\langle 0|\Big[\sum_{ij} \alpha_{ij}|i\rangle\langle j|\Big]|0\rangle\langle 1| \\
&+ p\ |0\rangle\langle 1|\Big[\sum_{ij} \alpha_{ij}|i\rangle\langle j|\Big]|1\rangle\langle 0| \\
&+ (1-p)\ |1\rangle\langle 1|\Big[\sum_{ij} \alpha_{ij}|i\rangle\langle j|\Big]|1\rangle\langle 1| \\
=& \alpha_{11}(1-p)\ |0\rangle\langle 0| + \alpha_{11}p\ |1\rangle\langle 1| \\
&+ \alpha_{22}p\ |0\rangle\langle 0| + \alpha_{22}(1-p)\ |1\rangle\langle 1|.
\end{aligned}$$

Using $\alpha_{11} := \alpha, \alpha_{22} = 1-\alpha$, we get $\mathcal{E}(\rho_X) = [\alpha(1-p) + (1-\alpha)p]\ |0\rangle\langle 0| + [\alpha p + (1-\alpha)(1-p)]\ |1\rangle\langle 1|$. The channel ignores the off-diagonal terms of ρ_X: it acts as a measurement on the computational basis followed by the classical binary symmetric channel.

d) *Now consider an arbitrary classical channel* $\mathbf{p}$ *from an n-bit space X to an m-bit space Y, defined by the conditional probabilities* $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ *as a map* $\mathcal{E}_\mathbf{p} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ *in the operator-sum representation.*

We generalize the previous result as

$$\begin{aligned}
\mathcal{E}_{\mathbf{p}}(\rho_X) &= \sum_{x,y} P_{Y|X=x}(y)\ |y\rangle\langle x|\rho_X|x\rangle\langle y| \\
&= \sum_{x,y} E_{x\to y}\rho_X E^\dagger x \to y, \quad E_{x\to y} = \sqrt{P_{Y|X=x}(y)}\ |y\rangle\langle x|.
\end{aligned}$$

To see that this works, take a classical state $\rho_X = \sum_x P_X(x)\ |x\rangle\langle x|$ as input,

$$\begin{aligned}
\mathcal{E}_{\mathbf{p}}(\rho_X) &= \sum_{x,y} P_{Y|X=x}(y)\ |y\rangle\langle x|\Big(\sum_{x'} P_X(x')\ |x'\rangle\langle x'|\Big)|x\rangle\langle y| \\
&= \sum_{x,y} P_{Y|X=x}(y)\ P_X(x)\ |y\rangle\langle y| \\
&= \sum_{y} P_y(y)\ |y\rangle\langle y|.
\end{aligned}$$

Exercise 8.2 Different Quantum Channels

Consider two single-qubit Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a CPTP map

$$\begin{aligned}
\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B) \\
\rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho,
\end{aligned}$$

which is called depolarizing channel*.*

a) *Find a Kraus representation for $\mathcal{E}_p$.*

For simplicity of notation, we denote the Pauli matrices by X, Y, Z.

Remembering that $X^2 = Y^2 = Z^2 = \mathbb{1}$, $XY = -YX = Z$, $YZ = -ZY = X$ and $ZX = -XZ = Y$, you can verify that

$$\mathbb{1} = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z).$$

From this follows the operator sum representation $\{M_x\}_x$,

$$M_1 = \sqrt{1 - \frac{3p}{4}}\ \mathbb{1}, \quad M_2 = \frac{\sqrt{p}}{2}X, \quad M_3 = \frac{\sqrt{p}}{2}Y, \quad M_4 = \frac{\sqrt{p}}{2}Z.$$

b) *What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? What is the physical interpretation of this?*

Using the result from part *a*) we have

$$\begin{aligned}
\mathcal{E}(\rho) &= \frac{p}{2}\mathbb{1} + (1-p)\ \rho \\
&= \frac{1}{2}(\mathbb{1} + (1-p)\ \vec{r}\cdot\vec{X})
\end{aligned}$$

Thus, points on a sphere with radius r are mapped to a smaller sphere with radius $(1-p)r$ — they get more mixed in that sense. In particular, pure states will not remain pure during this CPM.

c) *Find Kraus representations for the following qubit channels*

(i) *The dephasing channel:* $\rho \to \rho' = \mathcal{E}(\rho) = (1-p)\rho + p\,\mathrm{diag}(\rho_{00}, \rho_{11})$ *(the off-diagonal elements are annihiliated with probability p).*

The dephased output is the same as measuring the state in the standard basis: $\mathrm{diag}(\rho_{00}, \rho_{11}) = \sum_{j=0}^{1} P_j \rho P_j$ for $P_j = |j\rangle\langle j|$. Thus possible Kraus operators are $A_2 = \sqrt{1-p}\mathbb{1}$, $A_j = \sqrt{p}P_j$, $j = 0, 1$. But we can find a representation with fewer Kraus operators. Notice that $\sigma_z \rho \sigma_z = \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix}$. Thus $(\rho + \sigma_z \rho \sigma_z)/2 = \mathrm{diag}(\rho_{00}, \rho_{11})$ and $\rho' = \sum_{j=0}^{1} A_j \rho A_j^\dagger$ for $A_0 = \sqrt{1-p/2}\mathbb{1}$ and $A_1 = \sqrt{p/2}\sigma_z$.

(ii) *The amplitude damping (damplitude) channel, defined by the action* $|00\rangle \to |00\rangle$, $|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$.

From the unitary action we can read off the Kraus operators since $U|\psi\rangle|0\rangle = \sum_k A_k |\psi\rangle|k\rangle$. Therefore $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$.

Exercise 8.3 Classical capacity of the depolarizing channel

Consider the depolarizing channel we have treated in the exercise before that is described by the CPTP map

$$\begin{aligned} \mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B) \\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho. \end{aligned}$$

a) *Now we will see what happens when we use this quantum channel to send classical information. We start with an arbitrary input probability distribution* $P_X(0) = q, P_X(1) = 1-q$. *We encode this distribution in a state* $\rho_X = q\ |0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$. *Now we send* ρ_X *over the quantum channel, i.e., we let it evolve under* $\mathcal{E}_\mathbf{p}$. *Finally, we measure the output state,* $\rho_Y = \mathcal{E}_\mathbf{p}(\rho_X)$ *in the computational basis. Compute the conditional probabilities* $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Applying the map to this state results in

$$\begin{aligned} \mathcal{E}(\rho_X) &= \left(\frac{p}{2} + (1-p)q\right)\ |0\rangle\langle 0| + \left(\frac{p}{2} + (1-p)(1-q)\right)\ |1\rangle\langle 1| \\ &= P_Y(0)\ |0\rangle\langle 0| + P_Y(1)\ |1\rangle\langle 1|, \end{aligned}$$

so $P_Y(0) = \frac{p}{2} + (1-p)q, P_Y(1) = \frac{p}{2} + (1-p)(1-q)$. The conditional probabilities can be arranged in a transition matrix $(T)_{xy} = P_{Y|X=x}(y)$ as follows:

$$T = \begin{pmatrix} \frac{p}{2} + (1-p) & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} + (1-p) \end{pmatrix} = \begin{pmatrix} 1 - \frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & 1 - \frac{p}{2} \end{pmatrix}.$$

We obtained the binary symmetric channel, with $p' = p/2$.

b) *Maximize the mutual information over* q *to find the classical channel capacity of the depolarizing channel.*

The channel capacity of the binary symmetric channel, as has been shown in a previous exercise, is given by

$$C = 1 - H_{\text{bin}}(p/2), \quad H_{\text{bin}}(r) = -\left(r \log r + (1-r)\log(1-r)\right), \quad r \in [0,1].$$

c) *What happens to the channel capacity if we measure the final state in a different basis?*

Take an arbitrary basis $\left\{|\alpha\rangle, |\alpha^\perp\rangle\right\}$, where

$$|\alpha\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle, \quad |\alpha^\perp\rangle = \cos\left(\alpha + \frac{\pi}{2}\right)|0\rangle + \sin\left(\alpha + \frac{\pi}{2}\right)|1\rangle = -\sin\alpha|0\rangle + \cos\alpha|1\rangle.$$

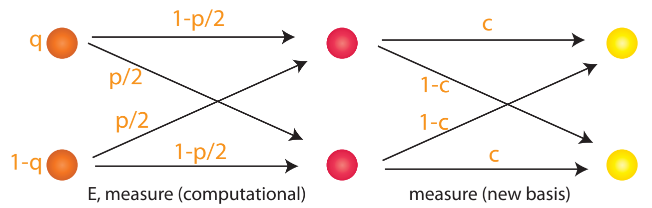

Figure 1: The result is a binary symmetric channel with $p' = 1 - c - p/2 + pc$.

Then

$$\begin{aligned} P_Y(\alpha) &= \mathrm{Tr}\left[|\alpha\rangle\langle\alpha|\ \mathcal{E}(\rho_X)\right] = \mathrm{Tr}\left[\begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}\begin{pmatrix} P_Y(0) & 0 \\ 0 & P_Y(1) \end{pmatrix}\right] \\ &= \cos^2(\alpha)P_Y(0) + \sin^2(\alpha)P_Y(1), \\ P_Y(\alpha^\perp) &= \mathrm{Tr}\left[|\alpha^\perp\rangle\langle\alpha^\perp|\ \mathcal{E}(\rho_X)\right] = \mathrm{Tr}\left[\begin{pmatrix} \sin^2\alpha & -\cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & \cos^2\alpha \end{pmatrix}\begin{pmatrix} P_Y(0) & 0 \\ 0 & P_Y(1) \end{pmatrix}\right] \\ &= \sin^2(\alpha)P_Y(0) + \cos^2(\alpha)P_Y(1). \end{aligned}$$

We can see this result in the following way: take $c = \cos^2(\alpha)$. Then " preparing $q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$, applying $\mathcal{E}_p$ and measuring in basis $\{|\alpha\rangle, |\alpha^\perp\rangle\}$" is equivalent to the concatenation of two binary symmetric channels (Fig. 1).

The final probability distributions are the same if we apply $\mathcal{E}_p$, measure in the computational basis, and then measure again in the new basis. This holds because $\mathcal{E}_p$ does not change the eigenbasis of the state, and is not necessarily true for a general TPCPM.

The capacity of the original channel is larger than the capacity of the concatenation of the two channels (because adding another channel just adds more noise, a fact otherwise known as the data processing inequality).