Eidgenössische Technische Hochschule Zürich
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Quantum Information Theory
Solutions 5

HS 13
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Exercise 5.1 Bloch sphere

We will see that density operators of two-level systems (qubits) can always be expressed as

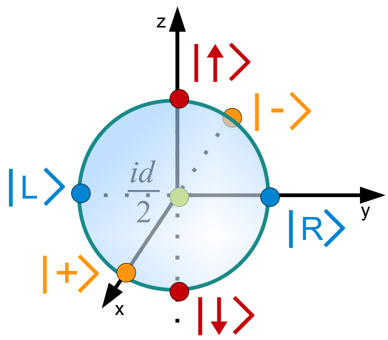
$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ *and* $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ *is the so-called Bloch vector, that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere.*

1. *Using* (1) *:*

   1) *Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases,* $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$ *and* $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$.

state	**density matrix**	**Bloch vector**	**in the figure**
$\frac{\mathbb{1}}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(0,0,0)$	green
$\|0\rangle$	$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$(0,0,1)$	red
$\|1\rangle$	$\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$(0,0,-1)$	red
$\|+\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$(1,0,0)$	yellow
$\|-\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$(-1,0,0)$	yellow
$\|\circlearrowright\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$	$(0,1,0)$	blue: $\|R\rangle$
$\|\circlearrowleft\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$	$(0,-1,0)$	blue: $\|L\rangle$



   2) *Find and diagonalise the states represented by Bloch vectors* $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ *and* $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.
   We have

$$\begin{aligned}
\rho_1 &= \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{2}, 0, 0\right) \cdot (\sigma_x, \sigma_y, \sigma_z)\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
&= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \Rightarrow \qquad \text{Eigenvalues: } \left\{\frac{1}{4}, \frac{3}{4}\right\},
\end{aligned}$$

$$
\begin{aligned}
\rho_2 &= \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot(\sigma_x, \sigma_y, \sigma_z)\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & 1 \\ 1 & \sqrt{2}-1 \end{pmatrix} \qquad \Rightarrow \qquad \text{Eigenvalues: } \{0, 1\}\,.
\end{aligned}
$$

The first Bloch vector lies inside the ball ($|\vec{r}_1 = \frac{1}{4}|$), and the state that it represents is mixed. The Bloch vector of the second state is on the surface of the sphere, and that state is pure.

2. *Show that the operator* ρ *defined in* (1) *is a valid density operator for any vector* $\vec{r}$ *with* $|\vec{r}| \leq 1$ *by proving it fulfills the following properties:*

   1) *Hermiticity:* $\rho = \rho^\dagger$.
      All Pauli matrices are Hermitian and the vector $\vec{r}$ is real, so the result comes from direct application of (1).
   2) *Positivity:* $\rho \geq 0$.
      The general form of a state given by (1) is

      $$\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix} \quad \Rightarrow \quad \text{Eigenvalues: } \left\{\frac{1-|\vec{r}|}{2}, \frac{1+|\vec{r}|}{2}\right\}. \tag{2}$$

      Since $0 \leq |\vec{r}| \leq 1$, the eigenvalues are non negative.
   3) *Normalisation:* $Tr(\rho) = 1$.
      From (2) we have that

      $$\mathrm{Tr}(\rho) = \frac{1-|\vec{r}|}{2} + \frac{1+|\vec{r}|}{2} = 1.$$

3. *Now do the converse: show that any two-level density operator may be written as* (1).

   One can always expand an operator A in an orthonormal basis $\{e_i\}_i$ as

   $$A = \sum_i (A, e_i) e_i,$$

   where the inner product (A, B) is defined as $\mathrm{Tr}(A^* B)$.

   The three Pauli matrices and the identity form a basis for 2×2 matrices, $\mathcal{B}$. However, this basis is not normalized. A normalised basis would be

   $$\mathcal{B}' = \left\{\frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}, \frac{\mathbb{1}}{\sqrt{2}}\right\}. \tag{3}$$

   We can expand any 2×2 matrix in this basis, and in particular any two-level density operator:

   $$\rho = \mathrm{Tr}(\rho\mathbb{1})\,\frac{\mathbb{1}}{2} + \sum_i \mathrm{Tr}(\rho\sigma_i)\,\frac{\sigma_i}{2} \tag{4}$$

   $$= \frac{\mathbb{1}}{2} + \frac{1}{2}\Big(\mathrm{Tr}(\rho\sigma_x), \mathrm{Tr}(\rho\sigma_y), \mathrm{Tr}(\rho\sigma_z)\Big)\cdot\Big(\sigma_x, \sigma_y, \sigma_z\Big) \tag{5}$$

   $$= \frac{\mathbb{1}}{2}\,(r_x, r_y, r_z)\cdot(\sigma_x, \sigma_y, \sigma_z) \tag{6}$$

Where we used the property of density operators $\mathrm{Tr}(\rho) = 1$. To obtain the bound $|\vec{r}| \leq 1$ we use the fact that for any density operator $\mathrm{Tr}(\rho^2) \leq 1$ (because all eigenvalues $\lambda_j \leq 1$ and $\sum \lambda_j = 1$) and get

$$\begin{aligned}
1 &\geq \mathrm{Tr}(\rho^2) \\
&= \mathrm{Tr}\Big(\left[\frac{\mathbb{1}}{2} + \sum_i r_i \frac{\sigma_i}{2}\right] \left[\frac{\mathbb{1}}{2} + \sum_i r_i \frac{\sigma_i}{2}\right] \Big) \\
&= \frac{1}{4}\mathrm{Tr}\Big(\left[1 + \sum_i {r_i}^2\right] \mathbb{1} \Big) \qquad \text{(because } \mathcal{B}' \text{ is an orthonormal basis)} \\
&= \frac{1}{2}\Big(1 + \sum_i {r_i}^2\Big) \\
1 &\geq \sum_i {r_i}^2.
\end{aligned}$$

4. *Check that the surface of the ball — the Bloch sphere — is formed by all the pure states.*

   For pure states, $\mathrm{Tr}(\rho^2) = 1$ and we can replace all "$\geq$" with "$=$" above, obtaining $|\vec{r}| = 1$.

5. *Discuss the analog of the Bloch sphere in higher dimensions. What can be said? For instance, where are the pure states?*

   `http://en.wikipedia.org/wiki/Bloch_sphere#A_generalization_for_pure_states` :)

Exercise 5.2 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{7}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then

$$H = \tfrac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{8}$$

1. *Show that H is unitary.*

   A matrix U is unitary when $U^\dagger U = \mathbb{1}$. In fact, $H^\dagger = H$, so we just need to verify that $H^2 = \mathbb{1}$, which is the case.

2. *What are the eigenvalues and eigenvectors of H?*

   Since $H^2 = \mathbb{1}$, its eigenvalues must be ± 1. If both eigenvalues were equal, it would be proportional to the identity matrix. Thus, one eigenvalue is $+1$ and the other -1. By direct calculation we can find that the (normalized) eigenvectors are

$$|\lambda_\pm\rangle = \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2}|0\rangle + \frac{1}{\sqrt{2(2 \pm \sqrt{2})}}|1\rangle \tag{9}$$

3. *What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?*

   The eigenbasis of $\sigma_{\hat{x}}$ is formed by the two states $|\hat{x}_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. From the form of H given in (8), it is clear that we can express H as

$$H = |\hat{x}_+\rangle\langle 0| + |\hat{x}_-\rangle\langle 1| \quad \text{or} \tag{10}$$
$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| \tag{11}$$

   The latter form follows immediately from the first since $H^\dagger = H$. Finally, we can express the $\sigma_{\hat{z}}$ basis $|0/1\rangle$ in terms of the $\sigma_{\hat{x}}$ basis as $|0\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_+\rangle + |\hat{x}_-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_+\rangle - |\hat{x}_-\rangle)$. Thus, if we replace $|0\rangle$ and $|1\rangle$ by these expressions in the equation for H we find

$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| = \frac{1}{\sqrt{2}}\left(|\hat{x}_+\rangle\langle \hat{x}_+| + |\hat{x}_-\rangle\langle \hat{x}_+| + |\hat{x}_+\rangle\langle \hat{x}_-| - |\hat{x}_-\rangle\langle \hat{x}_-|\right). \tag{12}$$

   Evidently, H has exactly the same representation in the $\sigma_{\hat{x}}$ basis! In retrospect, we should have anticipated this immediately once we noticed that H interchanges the $\sigma_{\hat{z}}$ and $\sigma_{\hat{x}}$ bases.

   For $\sigma_{\hat{y}}$, we can proceed differently. What is the action of H on the $\sigma_{\hat{y}}$ eigenstates? These are $|\hat{y}_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. Thus,

$$\begin{aligned}
H|\hat{y}_\pm\rangle &= \frac{1}{\sqrt{2}}\left(H|0\rangle \pm iH|1\rangle\right) && (13)\\
&= \tfrac{1}{2}\left(|0\rangle + |1\rangle \pm i|0\rangle \mp i|1\rangle\right) && (14)\\
&= \left(\frac{1 \pm i}{2}\right)|0\rangle + \left(\frac{1 \mp i}{2}\right)|1\rangle && (15)\\
&= \frac{1}{\sqrt{2}}e^{i\pm\frac{\pi}{4}}\left(|0\rangle + \left(\frac{1 \mp i}{1 \pm i}\right)|1\rangle\right) && (16)\\
&= \frac{1}{\sqrt{2}}e^{i\pm\frac{\pi}{4}}\left(|0\rangle \mp i|1\rangle\right) && (17)\\
&= e^{i\pm\frac{\pi}{4}}|\hat{y}_\mp\rangle && (18)
\end{aligned}$$

   Therefore, the Hadamard operation just swaps the two states in the basis (note that if we used a different phase convention for defining the $\sigma_{\hat{y}}$ eigenstates, there would be extra phase factors in this equation). So, $H = \begin{pmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix}$ in this basis.

4. *Give a geometric interpretation of the action of H in terms of the Bloch sphere.*

   All unitary operators on a qubit are rotations of the Bloch sphere by some angle about some axis. Since $H^2 = \mathbb{1}$, it must be a π rotation. Because the $\hat{y}$-axis is interchanged under H, the axis must lie somewhere in the $\hat{x}$-$\hat{z}$ plane. Finally, since H interchanges the $\sigma_{\hat{x}}$ and $\sigma_{\hat{z}}$ bases, it must be a rotation about the $\hat{m} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$ axis.