ETH
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Quantum Information Theory
Series 7

HS 13
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Have a look at a nice and easy paper for bedtime reading: http://iopscience.iop.org/1464-4266/5/3/357
It is about photonical implementation of a simple POVM measurement.

Exercise 7.1 Generalized Measurement by Direct (Tensor) Product

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle$ and the two systems interact via the unitary U_{AB} as follows:

$$\begin{aligned} |0\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B\right) \\ |1\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{6}}\left(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right) \end{aligned}$$

1. Calculate the measurement operators acting on A corresponding to a measurement on B in the canonical basis $|0\rangle, |1\rangle, |2\rangle$.
2. Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?
3. Suppose A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the state after a measurement, averaging over the measurement result?

Exercise 7.2 Unambiguous State Discrimination

Suppose that Bob has a state ρ that can either be ρ_1 and ρ_2, but he does not know which one. Bob wants to guess which state he has, and he wants to never guess wrong. He can achieve that, if he is allowed to not make a guess at all based on result of his measurement.

1. Bob's measurement surely has outcomes E_1 and E_2 corresponding to ρ_1 and ρ_2, respectively. Assuming the two states ρ_j are pure, $\rho_j = |\phi_j\rangle\langle\phi_j|$ for some $|\phi_j\rangle$, what is the general form of E_j such that $\Pr(E_j|\rho_k) = 0$ for $j \neq k$?
2. Can these two elements alone make up a POVM? Is there generally an inconclusive result $E_?$?
3. Assuming ρ_1 and ρ_2 are sent with equal probability, what is the optimal unambiguous measurement, i.e. the unambigous measurement with the smallest probability of an inconclusive result?

Exercise 7.3 Decompositions of Density Matrices

Consider a mixed state ρ with two different pure state decompositions

$$\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k| = \sum_{\ell=1}^{n} p_\ell |\phi_\ell\rangle\langle\phi_\ell|,$$

the former being the eigendecomposition so that $\{|k\rangle\}$ is an orthonormal basis.

1. Show that the probability vector $\vec{\lambda}$ *majorizes* the probability vector $\vec{p}$, which means that there exists a doubly stochastic matrix T_{jk} such that $\vec{p} = T\vec{\lambda}$. The defining property of doubly stochastic, or bistochastic, matrices is that $\sum_k T_{jk} = \sum_j T_{jk} = 1$.
   Hint: Observe that for a unitary matrix U_{jk}, $T_{jk} = |U_{jk}|^2$ is doubly stochastic.

2. The uniform probability vector $\vec{u} = (1/n, \ldots, 1/n)$ is invariant under the action of an $n \times n$ doubly stochastic matrix. Is there an ensemble decomposition of ρ such that $p_\ell = 1/n$ for all ℓ?
   Hint: Try to show that $\vec{u}$ is majorized by any other probability distribution.

Exercise 7.4 Broken Measurement

Alice and Bob share a state $|\Psi\rangle_{AB}$, and Bob would like to perform a measurement described by projectors P_j on his part of the system, but unfortunately his measurement apparatus is broken. He can still perform arbitrary unitary operations, however. Meanwhile, Alice's measurement apparatus is in good working order. Show that there exist projectors P_j' and unitaries U_j and V_j so that

$$|\Psi_j\rangle = (\mathbb{1} \otimes P_j)\,|\Psi\rangle = (U_j \otimes V_j)\left(P_j' \otimes \mathbb{1}\right)|\Psi\rangle.$$

(Note that the state is unnormalized, so that it implicitly encodes the probability of outcome j.) Thus Alice can assist Bob by performing a related measurement herself, after which they can locally correct the state.
Hint: Work in the Schmidt basis of $|\Psi\rangle$.