Exercise 1. ***Charge and momentum operators***

The quantized fields for a complex scalar and a fermion can be written as

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} \left(a_{\vec{p}}\, e^{-i\,px} + b^\dagger_{\vec{p}}\, e^{i\,px} \right)$$

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} \sum_{s=1,2} \left(b_{\vec{p},s}\, u(p,s) e^{-i\,px} + d^\dagger_{\vec{p},s}\, v(p,s) e^{i\,px} \right)$$

(a) Compute the (normal ordered) charge operator $Q = \int d^3\vec{x}\, i :\phi^\dagger \overleftrightarrow{\partial_0} \phi:$ for a complex scalar field and interpret the result.

(b) Compute the (normal ordered) charge operator $Q = \int d^3\vec{x}\ :\psi^\dagger \psi:$ for a fermion field. Show that the states $b^\dagger_{\vec{q}}|0\rangle$ and $d^\dagger_{\vec{q}}|0\rangle$ are eigenstates of Q and find the eigenvalues. How does this generalize to arbitrary states of the Fock space?

(c) Compute the momentum operator P^μ for a free fermion field.

(d) Show that P^μ commutes with Q and hence that $b^\dagger_{\vec{q}}|0\rangle$ and $d^\dagger_{\vec{q}}|0\rangle$ are also eigenstates of P^μ.

Exercise 2. ***Fermion propagator***

(a) Compute the Feynman propagator for fermions $i\, S_F(x-y) \equiv \langle 0| T\, \psi_\alpha(x)\, \bar{\psi}_\beta(y) |0\rangle$. What is the interpretation of the two terms from the time ordering?

(b) Show that $S_F(x-y)$ is a Green function for the Dirac equation.

Exercise 3. ***Gauge invariance of QED Lagrangian***

The Lagrangian of QED is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - e\bar{\psi}\, A\!\!\!/\psi$$

(a) Show that this Lagrangian is invariant under gauge transformations $\psi(x) \to e^{-ie\chi(x)}\psi(x)$, $A^\mu(x) \to A^\mu(x) + \partial^\mu \chi(x)$.

(b) How do the electric $\vec{E}$ and magnetic field $\vec{B}$ transform under gauge transformations?

(c) Show that the gauge freedom can be used to fix the potential such that it satisfies the conditions $A^0 = 0$ and $\vec{\nabla} \cdot \vec{A} = 0$.