Sheet X

Return by 5.12.2013

Question 1 [*Tensor product of* $\mathfrak{su}(3)$ *representations*]: Let V be the defining 3-dimensional representation of $\mathfrak{su}(3)$.

(a) Consider the tensor product $V \otimes V$ and decompose it into its symmetrical and antisymmetrical subspace. Show that each of these subspaces defines separately an irreducible representation of $\mathfrak{su}(3)$, and identify the relevant representation in terms of its highest weight. Compute the dimensions of these representations.

(b) Consider now the tensor product $V \otimes V \otimes V =: V^{\otimes 3}$. Find its totally symmetrical and totally anti-symmetrical subspace, and determine their dimensions. Assuming that they define by themselves two irreducible representations of $\mathfrak{su}(3)$, find the highest weights of the corresponding irreducible highest weight representations.

(c) Consider now the subspace of $V^{\otimes 3}$ spanned by the vectors

$$(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) + (\mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_k) - (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) - (\mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j) \ ,$$
$$(\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) + (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) - (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) - (\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j) \ ,$$

where $i, j, k \in \{1, 2, 3\}$, and the elements $\mathbf{e}_i$ form a basis for V. This subspace is the image of the Young symmetriser

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

on $V^{\otimes 3}$. Show that this space forms by itself an irreducible representation of $\mathfrak{su}(3)$, determine its weights, and hence identify the corresponding irreducible representation.

Hint: to show irreducibility you must check that, starting from the highest weight, the action of the 'lowering' operators does not generate other non-zero highest weight vectors.

(d) Repeat the analysis above for the subspace spanned by the vectors

$$(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) + (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) - (\mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_k) - (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) \ ,$$
$$(\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j) + (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) - (\mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j) - (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) \ ,$$

which is the image of the Young symmetriser

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

on $V^{\otimes 3}$.

(e) Deduce the full decomposition of $V^{\otimes 3}$ in terms of irreducible representations of $\mathfrak{su}(3)$.