Sheet VIII

Return by 21.11.2013

Question 1 [*Comparison between* $\mathfrak{sl}(2,\mathbb{R})$ *and* $\mathfrak{su}(1,1)$]: In this exercise we will consider the two three-dimensional Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(1,1)$. They are defined as subgroups of the general linear groups

$$\mathrm{SL}(2,\mathbb{R}) := \Big\{ M \in \mathrm{GL}(2,\mathbb{R}) \,:\, \det(M) = 1 \Big\} \,, \tag{1}$$

$$\mathrm{SU}(1,1) := \Big\{ M \in \mathrm{GL}(2,\mathbb{C}) \,:\, \det(M) = 1,\ M^\dagger \cdot I_{(1,1)} \cdot M = I_{(1,1)} \Big\} \,, \tag{2}$$

where

$$I_{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,.$$

a) Construct the corresponding real Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$, respectively. Compute their Killing forms and show that they are both semi-simple.

b) Show that the complexifications of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$ are isomorphic; that is, construct an isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$, considered as complex vector spaces.

Question 2 [*Special orthogonal and symplectic algebras*]: The special orthogonal group $\mathrm{SO}(N)$ is defined as

$$\mathrm{SO}(N) := \Big\{ M \in \mathrm{GL}(N,\mathbb{R}) \,:\, M^T \cdot M = I_N \text{ and } \det(M) = 1 \Big\} \,, \tag{3}$$

where I_N is the $N \times N$ identity matrix. On the other hand, the symplectic group $\mathrm{SP}(2N)$ is defined as

$$\mathrm{SP}(2N) := \Big\{ M \in \mathrm{GL}(N,\mathbb{R}) \,:\, M^T \cdot \Omega \cdot M = \Omega \Big\} \,, \tag{4}$$

where Ω is the $2N \times 2N$ real matrix (written in terms of $N \times N$ blocks)

$$\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \,. \tag{5}$$

Construct the corresponding Lie algebras $\mathfrak{so}(N)$ and $\mathfrak{sp}(2N)$, and calculate their dimensions.

Question 3 [*Embeddings of* $\mathfrak{su}(2)$ *in* $\mathfrak{su}(3)$]: Consider the Lie algebra $\mathfrak{su}(3)$, consisting of 3×3 antihermitian traceless matrices. For simplicity, consider the standard basis in terms of the eight Gell–Mann matrices[1]

$$\lambda_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{6}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{7}$$

[1]Note that this differs slightly from the usual definition of the Gell-Mann matrices: in particular, we included a factor of i in the definition of the matrices, and our definition of λ_8 is somewhat non-standard.

The aim is to identify the two inequivalent embeddings of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$, and to decompose in each case the adjoint representation of $\mathfrak{su}(3)$ into the representations of $\mathfrak{su}(2)$ under the adjoint action.

a) There is a 'naive' embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$ consisting of a direct identification of a set of $\mathfrak{su}(2)$ generators among the generators of $\mathfrak{su}(3)$. Find such an embedding, and decompose the adjoint representation of $\mathfrak{su}(3)$ in terms of representations of this $\mathfrak{su}(2)$.

   Hint: By construction the generators of $\mathfrak{su}(2)$ that you have chosen transform as the three dimensional (adjoint) representation under the action of $\mathfrak{su}(2)$. In order to work out the remaining decomposition, it is easier to work with the complexified $\mathfrak{sl}(2)$ generators, rather than $\mathfrak{su}(2)$. Thus you should construct the generators H, E, F you have seen in the lecture in terms of complex linear combinations of the $\mathfrak{su}(2)$ generators you have picked. Then find two different linear combinations of Gell-Mann matrices $M_k = \sum_i \alpha_i^k \lambda_i$, $k = 1, 2$ such that

$$\Big[E, M_k\Big] = 0 \ , \qquad \Big[H, M_k\Big] = M_k \ .$$

   Show that each of these two elements belongs to a distinct two-dimensional representation of $\mathfrak{su}(2)$; in the end, show that the residual one-dimensional subalgebra transforms trivially under $\mathfrak{su}(2)$.

b) For the other embedding consider the generators

$$\begin{aligned} D &= 2(\lambda_3 + \lambda_8) \ , \\ M_{12} &= \sqrt{2}(\lambda_2 + \lambda_7) \ , \\ \widehat{M}_{12} &= \sqrt{2}(\lambda_1 + \lambda_6) \ . \end{aligned}$$

   Show that they form the subalgebra $\mathfrak{su}(2)$. Decompose the adjoint representation of $\mathfrak{su}(3)$ in terms of representations of this $\mathfrak{su}(2)$.

   Hint: As before, construct the complexified generators H, E, F, and check that it is possible to find a linear combination of Gell-Mann matrices $N = \sum_i \beta_i \lambda_i$ such that

$$\Big[E, N\Big] = 0 \ , \qquad \Big[H, N\Big] = 4N \ .$$

   Show that N generates a five-dimensional irreducible representation of $\mathfrak{sl}(2)$ by acting repeatedly with F.