Sheet III

Return by 10.10.2013

Question 1 [*Eigenvalues of representations of finite groups*]: Consider a finite dimensional representation (V, ρ) of a finite group G. Show that the eigenvalues of $\rho(g)$ are roots of unity for any $g \in G$.

Question 2 [*Characters and irreducible representations*]: Recall that, given a representation W, the multiplicity with which a given irreducible representation V appears as a subrepresentation of W is

$$m_V^{(W)} = \langle \chi_W, \chi_V \rangle \ , \tag{1}$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined as

$$\langle \chi_A, \chi_B \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_A(g)}\, \chi_B(g) \ . \tag{2}$$

(a) Show that characters of inequivalent irreducible representations are mutually orthogonal.

(b) Show that a representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

(c) Using these results, show that the tensor product of any irreducible representation with a 1-dimensional representation is another irreducible representation. [*Hint:* Recall also the results of Question 1.]

Question 3 [*Representation theory of S_4 and characters*]:

(a) Find all the irreducible representations of S_4 and their characters.

  (i) Identify the conjugacy classes of S_4.

  (ii) Find two 1-dimensional representations.

  (iii) Recall the 4-dimensional *permutation representation*, which is defined by permuting the basis vectors e_1, e_2, e_3, e_4 according to

$$\rho_{\mathrm{p}}(\sigma) e_i = e_{\sigma(i)} \ .$$

  This representation is reducible. Find its irreducible subrepresentations.

  [*Hint:* Irreducibility can be shown using the character techniques developed in Question 2.]

  (iv) What must the dimensions of the remaining irreducible representations be? Construct one of them directly by tensoring two representations you have already found (cf. Question 2). Write down a character table and deduce from it the character of the missing representation.

*(v) Construct this last representation explicitly.

[*Hint:* This representation can be constructed from the standard 2-dimensional representation of S_3, upon defining the action of the transposition $\sigma_3 = (34)$ that is compatible with the relations of S_4, namely

$$\sigma_i \sigma_j = \sigma_j \sigma_i \;\; \forall \, |i-j| \geq 2 \, , \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \, , \quad \sigma_i^2 = e \, ,$$

where $\sigma_1 = (12)$ and $\sigma_2 = (23)$.]

(b) In (a) you should have found two 3-dimensional representations, which we may call ρ_3 and ρ_3'. Compute the decomposition of their tensor product $\rho_3 \otimes \rho_3'$ into irreducible components using character techniques (i.e., making use of the behaviour of characters under tensor products and direct sums of representations).