Exercise 1. ***Magnetic domain wall.***

We want to calculate the energy of a magnetic domain wall in the framework of the Ginzburg-Landau (GL) theory. Assuming translational symmetry in the (y, z)-plane, the GL functional in zero field reads

$$F[m, m'] = F_0 + \int dx \left\{ \frac{A}{2} m(x)^2 + \frac{B}{4} m(x)^4 + \frac{\kappa}{2} [m'(x)]^2 \right\}. \tag{1}$$

(a) Solve the GL equation with boundary conditions

$$m(x \to \pm\infty) = \pm m_0, \quad m'(x \to \pm\infty) = 0, \tag{2}$$

where m_0 is the magnetization of the uniform solution.

(b) First, find the energy of the uniformly polarized solution (no domain walls). Next, compute the energy of the solution with a domain wall compared to the uniform solution. Use the coefficients A, B and κ according to the expansion of the mean-field free energy of the Ising model (see Eqs. (5.78) and (5.83)). Finally, find the energy of a sharp step in the magnetization and compare it to the above results.

Exercise 2. ***Gaussian Fluctuations in the Ginzburg-Landau Model.***

Consider the Ginzburg-Landau model of the d-dimensional Ising model in presence of a magnetic field $H(\boldsymbol{r})$, introduced in chapter 5.4 of the lecture notes. Here, we only consider temperatures above the critical temperature T_c. In order to make the model exactly tractable, we assume that quartic fluctuations are negligible and ignore them. Therefore, the free energy functional for a given magnetization m and temperature T in d dimensions is given by

$$F(T, m, H) \; = \; \int d^d r \left\{ \frac{1}{2} A\, m(\boldsymbol{r})^2 - H(\boldsymbol{r})\, m(\boldsymbol{r}) + \frac{1}{2} \kappa \big[\boldsymbol{\nabla} m(\boldsymbol{r}) \big]^2 \right\} , \tag{3}$$

where $A = a\tau$, with $\tau = (T - T_c)/T_c$. For the calculations we assume our system to be a cube of side length L with periodic boundary conditions on m.

(a) Use the Fourier transform of the magnetization field,

$$m(\boldsymbol{r}) \; = \; \frac{1}{\sqrt{L^d}} \sum_{\boldsymbol{q}} m_{\boldsymbol{q}}\, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}} , \tag{4}$$

and compute the energy functional $F(T, m)$ in the transformed coordinates $\{m_{\boldsymbol{q}}\}$. Which values of $\boldsymbol{q}$ are allowed in the sum and which values of $\boldsymbol{q}$ are independent? Note that $m(\boldsymbol{r})$ is real and interpret its implication on the $m_{\boldsymbol{q}}$.

(b*) The calculation of the canonical partition function,

$$Z(T) \; = \; \int \mathcal{D}m\, \mathrm{e}^{-F(T,m)/k_B T} , \tag{5}$$

is rather involved. If you have time to spare, show that $Z(T)$ is equal to

$$Z(T) = \prod_{|q|<\Lambda} \sqrt{\frac{2\pi}{\beta X_{\boldsymbol{q}}}} \exp\left\{ \frac{|H_{\boldsymbol{q}}|^2}{2k_B T\,(A + \kappa \boldsymbol{q}^2)} \right\} , \tag{6}$$

by using Gaussian integration. Otherwise, proceed directly to point (c) using this result.

Hints. Argue that the finite number of degrees of freedom (finite lattice spacing) of our Ising model introduces a momentum cutoff Λ*, which is crucial to regulate the otherwise ill-defined integrals (cf. Debye wave vector for phonons).*

Rewrite the functional measure $\mathcal{D}m$ *according to*

$$\mathcal{D}m = \prod_{\boldsymbol{q}} dm_{\boldsymbol{q}}\, dm_{-\boldsymbol{q}} \,. \tag{7}$$

Why do we use $dm_{\boldsymbol{q}}\, dm_{-\boldsymbol{q}}$ *?*

You can also use the fact the measure $\mathcal{D}m$ *in Fourier space can be expressed as*

$$\mathcal{D}m = dm_0 \cdot \prod_{q \in \mathcal{A}^+} \sqrt{2}d(m'_{\boldsymbol{q}})\, \sqrt{2}d(m''_{\boldsymbol{q}}), \qquad \textit{with } m_{\boldsymbol{q}} = m'_{\boldsymbol{q}} + i\, m''_{\boldsymbol{q}} \,, \tag{8}$$

where $\mathcal{A}^+$ *is a choice of half the set of allowed momenta* $\boldsymbol{q} \neq 0$*, such that* $\boldsymbol{q} \in \mathcal{A}^+ \Leftrightarrow -\boldsymbol{q} \notin \mathcal{A}^+$*.*

(c) Determine the free energy $F(T) = -k_B T \log Z(T)$.

Compute the specific heat c_V in the thermodynamic limit $L \to \infty$ for vanishing external field ($H(\boldsymbol{r}) \equiv 0$). Study its behavior for different dimensions d near the critical temperature where $\tau = 0$. Compare the critical exponent of c_V with the mean field result of section 5.4.2 of the lecture notes.

(d) Derive an expression for the magnetic susceptibility, defined as the negative second derivative of the free energy with respect to the external field H in the limit of vanishing field, i.e.

$$\chi(T) = -\left.\frac{\partial^2 F(T)}{\partial H^2}\right|_{H=0} \,. \tag{9}$$

What is the critical exponent of χ? Compare the result with the mean field result of section 5.2.2 of the lecture notes, Eq. (5.28).