Exercise 1. ***Quantum rotor in a magnetic field***

Consider a lattice of N quantum rotors. Each rotor is independent and has a momentum of inertia $I = mR^2$. It is described by the following Hamiltonian:

$$H = \frac{\mathbf{L}^2}{2mR^2} = \frac{\mathbf{L}^2}{2I} . \tag{1}$$

(a) Calculate the (canonical) partition function of the system of N rotors. Determine the entropy, the internal energy, the projection of the angular moment along the z direction and the heat capacity. Compute them numerically and study the high and low temperature limits. It is useful to define θ_{rot} by $k_B\theta_{\text{rot}} = 1/I$.

Hint. If $f^{(n)}(\infty) \to 0, \forall n \in \mathbb{N}$ then the Euler–Maclaurin formula could be simplified to:

$$\sum_{l=0}^{\infty} f(l) = \int_0^{\infty} dl f(l) + \frac{1}{2} f(0) - \sum_{k=2}^{\infty} \frac{(-1)^k b_k}{(k)!} f^{k-1}(0) + R_\infty \tag{2}$$

where R_∞ is a small correction and b_k are the Bernoulli numbers $b_2 = 1/6, b_3 = 0, b_4 = -1/30, \cdots$.
http://people.csail.mit.edu/kuat/courses/euler-maclaurin.pdf

Now add a magnetic field that couples to the angular momentum as:

$$H' = -\gamma \mathbf{B} \cdot \mathbf{L} . \tag{3}$$

(b) What is the effect of the magnetic field? Determine the entropy, the internal energy, the projection of the angular moment along the z direction and the heat capacity. Compute them numerically and study the high and low temperature limits. It is useful to define θ_{mag} by $k_B\theta_{\text{mag}} = \gamma B_z$.

Exercise 2. ***Ideal fermionic quantum gas in a harmonic trap***

In this exercise we study the fermionic spinless ideal gas confined in a three-dimensional harmonic potential and compare it with classical case (for the results of the classical case see Exercise Sheet 1). The energy states of the gas are given by

$$E_{\mathbf{a}} = \hbar\omega(a_x + a_y + a_z) \tag{4}$$

where we neglect the zero point energy $E_0 = 3\,\hbar\omega/2$. The occupation number of the oscillator modes of the state $E_{\mathbf{a}}$ is given by $n_{\mathbf{a}}$ where $\mathbf{a} = (a_x, a_y, a_z)$ with $a_i \in \{0, 1, 2, ...\}$.

(a) Consider the high-temperature, low-density limit ($z \ll 1$). Derive the grand canonical partition function $\mathcal{Z}_f$ of this system and compute the grand potential Ω_f. Show that

$$\Omega_f \propto f_4(z) , \tag{5}$$

where the function $f_s(z)$ is defined as

$$f_s(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^s} . \tag{6}$$

(b) Derive the internal energy U and the average particle number $\langle N \rangle$. In order to get U in terms of N (instead of dealing with the chemical potential), introduce the parameter

$$\rho \equiv \left(\frac{\hbar \omega N^{1/3}}{k_B T} \right)^3 , \tag{7}$$

and relate it to z using the high-temperature, low-density expansion of $\langle N \rangle$. Interpret the condition $\rho \ll 1$.

Then, expand U up to second order in ρ, relating it to N .

(c) Compute the specific heat C. Which quantity has to be fixed in order to do this?

(d) Compute the isothermal compressibility κ_T .

(e) Interpret your results for U, C, and κ_T by comparing them with the corresponding results for the classical Boltzmann gas. How do the quantum corrections influence the fermionic system?