Exercise 1. ***The Classical Ideal Paramagnet Reloaded.***

Consider a lattice of N noninteracting particles, each possessing a magnetic moment $\vec{m}_i$ of fixed magnitude m which can point in any spacial direction. (This changes from last week's exercise, where $m_i = \pm m$.) The Hamiltonian is, as you might have guessed,

$$\mathcal{H} = -\sum_i \vec{m}_i \cdot \vec{H} \ , \tag{1}$$

where $\vec{H}$ is the externally applied magnetic field, assumed homogeneous and in the Z direction.

(a) Calculate the canonical partition function Z of the system.

(b) Calculate the free energy F, internal energy U and heat capacity C. Discuss the limiting cases where $k_B T \ll mH$ and $k_B T \gg mH$. Calculate the entropy S in those cases.

(c) If M_z is the thermodynamic variable corresponding to magnetization, show that

$$M_z = -\left(\frac{\partial F}{\partial H_z}\right)_{T,N} \ , \tag{2}$$

if the paramagnet obeys the Curie law $M_z = K\, H_z/T$ for some constant K.

Hint. Remember that in the thermodynamics of magnetic systems, H and M replace respectively variables p and V as conjugate variables.

(d) The magnetization in statistical mechanics is given by $M_z = \sum_i m_i^z$. Show explicitly that

$$\langle M_z \rangle = -\frac{\partial F}{\partial H_z} \ . \tag{3}$$

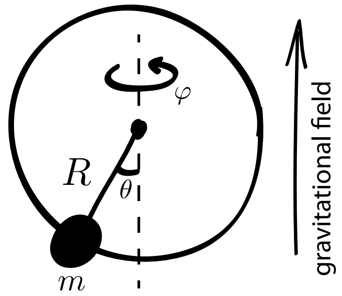
Calculate the value of $\langle M_z \rangle$. In which regime does the system obey Curie's law?

(e) Calculate the fluctuations $\langle M_z^2 \rangle - \langle M_z \rangle^2$ and relate them to the magnetic susceptibility $\chi_{zz} = (\partial M_z / \partial H_z)$.

Exercise 2. ***Rigid Pendulums.***

We will now consider a lattice of N classical rigid rotors. Each rotor is independent, is free to point in any spatial direction and has a moment of inertia $I = mR^2$. Its Hamiltonian is

$$\mathcal{H} = \frac{1}{2I}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) \ . \tag{4}$$



(a) Calculate the (canonical) partition function of the system of N rotors. Calculate the internal energy and the heat capacity. Study the regimes $T \to 0$ and $T \to \infty$.

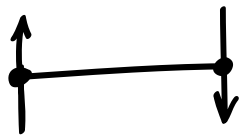
We now immerse the N rotors into a gravitational field with potential $V = mg\, x_{i,z} = -mgR\cos\theta_i$.

(b) Determine the partition function and compare it with the partition function of Exercise **??**. Calculate the free energy, internal energy and heat capacity of the system. Discuss the limits $T \to 0$ and $T \to \infty$.

Exercise 3. ***Independent Dimers in a Magnetic Field. Quantum vs Ising.***

We consider a system of N independent dimers of two spins, $s = 1/2$, described by the Hamiltonian

$$\mathcal{H}_0^{\text{quantum}} = J \sum_i \left(\vec{S}_{i,1} \cdot \vec{S}_{i,2} \right), \tag{5}$$



where i is the dimer index and $m = 1, 2$ denotes the spin state along z direction ($|\uparrow\rangle$ or $|\downarrow\rangle$). For simplicity, we use $\hbar = 1$. To this quantum system corresponds a classical Ising dimer, described by:

$$\mathcal{H}_0^{\text{Ising}} = \frac{1}{2} J \sum_i \left(\sigma_{i,1} \cdot \sigma_{i,2} - \frac{1}{2} \right), \tag{6}$$

where $\sigma_{i,m} = \pm 1$. The spins are aligned along the z axis. We will use eigenstates and eigenenergies to denote also the classical states and energies.

(a) What are the eigenstates and the eigenenergies of a single dimer for the two cases?

(b) For both cases consider the macroscopic system and determine the Helmholtz free energy, the entropy, the internal energy and the specific heat as a function of temperature and N. Discuss the limit $T \to 0$ and $T \to \infty$ for both signs of J (antiferromagnetic and ferromagnetic case).

Note: The following exercises are left for the fun of the interested students.

(c*) We now apply a magnetic field along z direction leading to an additional term in the Hamiltonian,

$$\mathcal{H}_{\text{mag}}^{\text{quantum}} = -g\mu_B H \sum_{i,m} S_{i,m}^z \tag{7a}$$

$$\mathcal{H}_{\text{mag}}^{\text{Ising}} = -g\mu_B H \sum_{i,m} \frac{\sigma_{i,m}}{2}. \tag{7b}$$

How do the eigenenergies change? Sketch the energies with respect to the applied field H, the partition functions and determine the ground state for both cases. For the antiferromagnetic case you should notice a critical field. What differences do you notice between the classical and quantum system when the the critical field is reached? For the quantum case discuss in this context the entropy per dimer in the limit $T \to 0$.

(d*) Calculate the magnetization m for the two cases. In which limit are they the same? Moreover compute the magnetic susceptibility χ for the quantum case and discuss its dependence on H for different temperatures.